68. A Note on Morita's P-spaces

By Vincent J. MANCUSO
St. John's University, Jamaica, New York, U. S. A.

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1. Introduction. K. Morita [10] introduced the notion of a P-space and demonstrated its importance in the theory of product spaces. The purpose of this note is to prove some results about P-spaces which will have application in homotopy extension. Let $A \subset X$ be closed and $f: X \to Y$ continuous. If, in the free union $X+Y$, we identify $a \in A$ with $f(a) \in Y$, we obtain a quotient space Z called the *adjunction space* of X and Y via the map f[3, p. 127]. A normal space X is called *totally normal* if every open subset G of X can be covered by a family locally finite in G, of open F_σ sets of X[1]. We will prove the following theorems:

Theorem 1. *If X and Y are normal P-spaces, the adjunction space Z of X and Y is a normal P-space.*

Theorem 2. *If X is a totally normal P-space and Y is a compact metric space, $X \times Y$ is a totally normal P-space.*

Actually, Theorem 2 will follow from the slightly more general:

Theorem 2′. *If X is totally normal and countably paracompact, and Y is compact metric, $X \times Y$ is totally normal.*

Theorem 3. *An open subspace of a totally normal P-space is a (normal) P-space.*

Remark 1. The compactness of Y in Theorem 2′ cannot be dropped since Michael [9] has given an example of a hereditarily paracompact (and hence totally normal and countably paracompact) space such that its product with a separable metric space is not normal. We are therefore led to the following question:

Question 1. If X is a totally normal P-space and Y is a metric space, is $X \times Y$ totally normal? Note that the normality of $X \times Y$ is assured since X is a normal P-space [10, Theorem 4.1]. In view of [11, Theorem 2], it would be sufficient to show that $X \times Y$ is hereditarily countably paracompact.

In proving Theorem 1, we will use the closed set dual of the definition of a P-space given in [10].

Definition 1. Let $\mathfrak{m}$ be a cardinal number ≥ 1. X is a *$P(\mathfrak{m})$-space* if for any set Ω of power $\mathfrak{m}$ and for any family $\{F(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ of closed sets of X such that $F(\alpha_1, \cdots, \alpha_i) \supset F(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$ for each sequence $\alpha_1, \alpha_2, \cdots$, there exists a family of open sets $\{G(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ satisfying $F(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i)$ for each sequence $\alpha_1, \alpha_2, \cdots$ and $\bigcap\{G(\alpha_1, \cdots \cdots, \alpha_i); i=1,2, \cdots\}=\emptyset$ for any sequence $\{\alpha_i\}$ such that $\bigcap\{F(\alpha_1, \cdots \alpha_i); i=1,2, \cdots\}=\emptyset$. X is called a *P-space* if X is a $P(\mathfrak{m})$-space for each $\mathfrak{m} \geq 1$.

Remark 2. By Ishikawa [7], a $P(1)$-space is countably metacompact, and by Dowker [2], a normal space is a $P(1)$-space if and only if it is countably paracompact.

2. Proofs of theorems. At this point, we will assume the reader is familiar with the salient features of the adjunction space as, for example, those given in [3, Theorem 6.3, p. 128], or in [5]. The proof of Theorem 1 now follows:

Since Z is normal [5, Lemma 3.3], we need only show that Z is a $P(\mathfrak{m})$-space for each $\mathfrak{m} \geq 1$. Let $\{F(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ be a family of closed sets of Z, with Ω of power $\mathfrak{m}$, such that $F(\alpha_1, \cdots, \alpha_i) \supset F(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$ for each sequence $\alpha_1, \alpha_2, \cdots$. Then since Y is a $P(\mathfrak{m})$-space, there is a family $\{V(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ of open subsets of Y satisfying:

(1) $$F(\alpha_1, \cdots, \alpha_i) \cap Y \subset V(\alpha_1, \cdots, \alpha_i)$$

for each sequence $\alpha_1, \alpha_2, \cdots$ and,

(2) $\bigcap\{V(\alpha_1, \cdots, \alpha_i); i=1,2, \cdots\}=\emptyset$ for any sequence $\{\alpha_i\}$ such that $\bigcap\{F(\alpha_1, \cdots, \alpha_i) \cap Y; i=1,2, \cdots\}=\emptyset$.

Since Y is normal, there exists a family $V'(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ of open subsets of Y such that

(3) $$F(\alpha_1, \cdots, \alpha_i) \cap Y \subset V'(\alpha_1, \cdots, \alpha_i) \quad \text{and}$$

(4) $$\mathrm{Cl}(V'(\alpha_1, \cdots, \alpha_i)) \subset V(\alpha_1, \cdots, \alpha_i) \quad (\mathrm{Cl}=\text{closure})$$

for each sequence $\alpha_1, \alpha_2, \cdots$. Moreover, we may assume that

(5) $$V'(\alpha_1, \cdots, \alpha_i) \supset V'(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$$

for each sequence $\alpha_1, \alpha_2, \cdots$.

Let $k=p|X$ where $p: X+Y \to Z$ is the natural projection. Now set $K(\alpha_1, \cdots, \alpha_i)=k^{-1}[F(\alpha_1, \cdots, \alpha_i) \cup \mathrm{Cl}(V'(\alpha_1, \cdots, \alpha_i))]$. Then

$$\{K(\alpha_1, \cdots, \alpha_i); \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$$

is a family of closed sets of X, and by (5),

(6) $$K(\alpha_1, \cdots, \alpha_i) \supset K(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$$

for each sequence $\alpha_1, \alpha_2, \cdots$.

Since X is a $P(\mathfrak{m})$-space, there exists a family $\{H(\alpha_1 \cdots, \alpha_i); \alpha_1, \cdots \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ of open subsets of X satisfying:

(7) $$K(\alpha_1, \cdots, \alpha_i) \subset H(\alpha_1, \cdots, \alpha_i)$$

for each sequence $\alpha_1, \alpha_2, \cdots$ and,

(8) $\bigcap\{H(\alpha_1, \cdots, \alpha_i); i=1,2, \cdots\}=\emptyset$ for any sequence $\{\alpha_i\}$ such that $\bigcap\{K(\alpha_1, \cdots, \alpha_i); i=1,2, \cdots\}=\emptyset$.

Let $G(\alpha_1, \cdots, \alpha_i)=k(H(\alpha_1, \cdots, \alpha_i)-A) \cup V'(\alpha_1, \cdots, \alpha_i)$. Using (7), we can argue as in [5, p. 377] to show that each $G(\alpha_1, \cdots, \alpha_i)$ is open in Z. Now writing

$F(\alpha_1, \cdots, \alpha_i)=[F(\alpha_1,\cdots,\alpha_i)\cap Y]\cup[F(\alpha_1,\cdots,\alpha_i)\cap Z-Y]$ and using (3) and the fact that $k|X-A$ is an embedding, we get

$$F(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i)$$

for each sequence $\alpha_1, \alpha_2, \cdots$.

Now suppose $\bigcap\{F(\alpha_1\cdots, \alpha_i); i=1,2,\cdots\}=\emptyset$ for some sequence $\{\alpha_2\}$. By (2) and (4), $\bigcap\{\mathrm{Cl}(V'(\alpha_1, \cdots, \alpha_i); i=1,2,\cdots\}=\emptyset$ and so $\bigcap\{K(\alpha_1, \cdots, \alpha_i); i=1, 2, \cdots\}=\emptyset$ for $\{\alpha_i\}$. By (8), $\bigcap\{H(\alpha_1, \cdots, \alpha_i); i=1,2,\cdots=\emptyset$ for $\{\alpha_i\}$. Again, since $k|X-A$ is an embedding, $\bigcap\{G(\alpha_1, \cdots, \alpha_i); i=1,2,\cdots\}=\emptyset$ for $\{\alpha_i\}$, and the proof is complete.

Proof of Theorem 2′. $X\times Y$ is normal by [2, Lemma 3]. Let G be open in $X\times Y$ and let $\mathfrak{B}=\{B_i; i\in N\}$ be a countable base for Y. For each $i\in N'\subset N$, there exists H_i open in X such that $H_i\times B_i\subset G$ and $G=\bigcup\{H_i\times B_i; i\in N'\}$. Let $\mathcal{H}=\bigcup\{H_i; i\in N'\}$. It follows easily from Theorem 1.3 in [10] with $m=\aleph_0$, that X is hereditarily countably paracompact. $\mathcal{H}$ is therefore countably paracompact and normal, and so there exists a locally finite (in $\mathcal{H}$) open refinement $\mathcal{V}=\{V_i\}$ such that $V_i\subset H_i$ for each $i\in N'$. Since X is totally normal, each $V_i=\bigcup\{W_{i\alpha}; \alpha\in\Omega_i\}$ where $\{W_{i\alpha}; \alpha\in\Omega_i\}$ is a collection, locally finite in V_i, of open F_σ sets of X.

For each $i\in N'$ and $\alpha\in\Omega_i$, let $C_{i\alpha}=W_{i\alpha}\times B_i$. Clearly, each $C_{i\alpha}$ is an open F_σ set of $X\times Y$, and $G=\bigcup\{C_{i\alpha}; \alpha\in\Omega_i, i\in N'\}$. We contend that $\{C_{i\alpha}; \alpha\in\Omega_i, i\in N'\}$ is locally finite in G: Let $(x,y)\in G$. There exists a neighborhood N_x of x in $\mathcal{H}$ (and hence in X) such that N_x meets at most $V_{i_1}, \cdots, V_{i_n}$. If $x\in V_{i_j}$, there exists a neighborhood N_{i_j} of x in V_{i_j} (and hence in X) such that N_{i_j} meets at most finitely many members of the family $\{W_{i_j\alpha}; \alpha\in\Omega_{i_j}\}$. Then $N=\bigcap\{N_{i_j}; x\in N_{i_j}\}$ is a neighborhood of x in X. Let $B_i(y)$ be any member of $\mathfrak{B}$ containing y. It follows that $(N_x\cap N)\times B_i(y)$ is a neighborhood of (x,y) in G which meets at most finitely many of $\{C_{i\alpha}; \alpha\in\Omega_i, i\in N'\}$, and this completes the proof.

Proof of Theorem 2. Since X is a P-space, $X\times Y$ is normal and since Y is compact, $X\times Y$ is a P-space [10, Corollary 3.5]. By Remark 2 and Theorem 2′, $X\times Y$ is totally normal.

Observe that with the hypothesis of Theorem 2′, $X\times Y$ is countably paracompact [2, Theorem 1] and so $X\times Y$ is hereditarily countably paracompact [10, Theorem 1.3].

Proof of Theorem 3. We will modify the technique used by Hodel [6]. Let G be an open subspace of a totally normal P-space X. Then $G=\bigcup\{G_\alpha; \alpha\in\Omega\}$ where $\{G_\alpha; \alpha\in\Omega\}$ is a family, locally finite in G, of open F_σ sets of X. Each G_α is a normal $P(m)$-space [10, Corollary 3.7]. Since G is normal, there exists an open refinement $\{V_\alpha;$

$\alpha \in \Omega\}$ of $\{G_\alpha; \alpha \in \Omega\}$ such that $\mathrm{Cl}(V_\alpha) \subset G_\alpha$ for $\alpha \in \Omega$ (closure is taken in G). Applying Corollary 3.7 again, and then Theorem 3.6 of [10], we get that G is a $P(\mathfrak{m})$-space. This completes the proof.

Remark 3. Theorem 3 also holds if P-space is everywhere replaced by M-space [10, p. 379]. A very similar argument works except that the crucial results needed on M-spaces are to be found in [8] and [13].

Question 2. Is every subspace of a totally normal P-space a P-space? Using Theorem 4.1 in [10], it is easy to show that an affirmative answer to Question 1 implies an affirmative answer to Question 2. However, another method is available. Hodel's technique [6, Theorem 1] would also yield an affirmative answer if we could prove the following: If every open subspace of X is a P-space, every subspace of X is a P-space. The same question appears to be open for M-spaces.

Example 1. Let $X = \beta Y$, the Stone-Cech compactification of Y where Y is Michael's example [9]. Clearly, X is a normal P-space but Y is a hereditarily paracompact subspace which is not a P-space.

Example 2. Let $X = w(Y)$, the Wallman compactification of Y where Y is Frolik's example [4]. Y is a Hausdorff space which is not countably compact and yet every point-finite open cover has a finite subcover. It follows that Y could not be countably metacompact. Therefore, X must have an open subspace which is not countably metacompact and hence not a P-space (see Remark 2). This example shows that an open subspace of a P-space X need not be a P-space.

Note that X of Example 2 is not normal. As yet the author has not found an example of an open subspace of a normal P-space which is not a P-space. Such an example can be found if there exists a completely regular T_1 space X which is not countably metacompact, for then, as above, βX must have an open subspace which is not a P-space.

Example 3. Let X be the Tychonoff plank, $[0, \Omega] \times [0, \omega]$, and let G be the open subspace $X - \{\Omega, \omega\}$. Clearly, X is a normal M-space and yet it is shown in [12] that G is not an M-space (but G is a P-space).

3. Applications to homotopy extension. We will merely state the theorems in this section, since their proofs will be given elsewhere.

Let $I = [0, 1]$ and $C = X \times \{0\} \cup A \times I$.

Theorem 4. *Let X be an ANR(normal P-space) and A a closed G_δ in X such that A is also an ANR(normal P-space). Then any continuous $f: C \to Y$ has a homotopy extension $F: X \times I \to Y$.*

Theorem 5. *Let A be closed in a totally normal P-space X such that* $\mathrm{Ind}(X - A) \leq n$. *Let Y be complete separable metric and LC^n. Then any continuous $f: C \to Y$ has a homotopy extension $F: X \times I \to Y$.*

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