53. *Properties of Ergodic Affine Transformations of Locally Compact Groups. I*

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1. Introduction. Let G be a locally compact group. An affine transformation S of G is a one-to-one continuous transformation of G onto itself which is of the form $S(x)=a\cdot T(x)$, where a is an element of G and T is a continuous isomorphism of G onto itself. In his book, Lectures on Ergodic Theory [1], Halmos has raised a question: Can an automorphism of a locally compact but non-compact group be an ergodic measure preserving transformation? Recently Rajagopalan and Schreiber [3] have answered his question negatively, i.e., if G is a locally compact group which has an ergodic continuous automorphism with respect to a Haar measure on G then G is compact.

Then the following question has become of interest to the author: Can an affine transformation of a locally compact but non-compact group be an ergodic left Haar measure preserving transformation?

The aim of this paper is to study some properties of an ergodic affine transformation of a locally compact group and to give an answer to the above question. We shall prove the followings below:

(1) An affine transformation S of a locally compact group G which is not *bi*-continuous can not be ergodic with respect to a left Haar measure on G.

(2) An affine transformation S of a locally compact group G which is not left Haar measure preserving can not be ergodic with respect to a left Haar measure on G.

(3) If G is a locally compact totally disconnected non-discrete group which has an ergodic affine transformation S with respect to a left Haar measure on G then G is compact.

2. Properties of ergodic affine transformations.

Theorem 1. *Let G be a locally compact group with a left Haar measure μ. Suppose $S(x)=a\cdot T(x)$ is an affine transformation of G which is not bi-continuous. Then S is not ergodic with respect to μ.*

Proof. Since S is not *bi*-continuous, T is not *bi*-continuous. Thus there exists an open σ-compact subgroup H of G such that $T(H)\subset H$ and $T^{-1}(H)$ is not σ-compact by [2, Lemma 1].

Case I. Let there exist a positive integer n for which $S^{-n}(H)\supset H$. Let p be the smallest positive integer such that $S^{-p}(H)\supset H$. Then it is easy to see that if m is a positive integer such that $S^{-m}(H)\supset H$ then $m=kp$ for some positive integer k. Since $S^{-p}(H)$ is not σ-compact, there exists an element x_1 in $S^{-p}(H)$ such that $(x_1H)\cap H=\phi$. Since x_1H is σ-compact and S is continuous, $\bigcup_{n=0}^{\infty} S^n(x_1H)$ is σ-compact. So there exists an element x_2 in $S^{-p}(H)$ such that $(x_2H)\cap H=(x_2H)\cap\left[\bigcup_{n=0}^{\infty} S^n(x_1H)\right]=\phi$. Since x_1H is open, $\bigcup_{n=1}^{\infty} S^{-n}(x_1H)$ is open. Hence the set

$$E=\bigcup_{n=-\infty}^{\infty} S^n(x_1H)$$

is a Borel set and clearly $S^{-1}(E)=E$ and $\mu(E)>0$ since E has non-void interior.

If $(x_2H)\cap E\neq\phi$ then there exists a positive integer n for which $(x_2H)\cap S^{-n}(x_1H)\neq\phi$. So $S^{-p}(H)\cap S^{-(n+p)}(H)\neq\phi$, hence $H\subset S^{-p}(H)\subset S^{-(n+p)}(H)$, thus $n+p=kp$ for some $k\geqq 2$. Since this is impossible from the choice of x_1 and x_2 and the fact that $S^{-p}(H)\supset H$, x_2H and E are disjoint. Therefore $\mu(G\cap E^c)\geqq\mu(x_2H)>0$.

Case II. Let $S^{-n}(H)\cap H=\phi$ for all positive integers n. Then $S^m(H)\cap S^n(H)=\phi$ for $m\neq n$. Since $S^{-1}(H)$ is not σ-compact, there exist two elements x_1 and x_2 in $S^{-1}(H)$ such that $(x_1H)\cap(x_2H)=\phi$. The set

$$F=\bigcup_{n=-\infty}^{\infty} S^n(x_1H)$$

is a Borel set such that $S^{-1}(F)=F$, $\mu(F)>0$ and $\mu(G\cap F^c)\geqq\mu(x_2H)>0$.

The proof is complete.

Theorem 2. *Let G be a locally compact group with a left Haar measure μ. Suppose $S(x)=a\cdot T(x)$ is an affine transformation of G which is not μ measure preserving. Then S is not ergodic with respect to μ.*

Proof. By Theorem 1, we may assume that S is *bi*-continuous. So T is *bi*-continuous, whence there exists a constant $\delta>0$ such that $\mu(T(E))=\delta\mu(E)$ for all Borel sets E by uniqueness of left Haar measure. Therefore $\mu(S(E))=\mu(a\cdot T(E))=\mu(T(E))=\delta\mu(E)$. Since $S^{-1}(x)=T^{-1}(a^{-1})\cdot T^{-1}(x)$, we have $\mu(S^{-1}(E))=\mu(T^{-1}(E))=\delta^{-1}\mu(E)$. Since S is not μ measure preserving, it follows $\delta\neq 1$.

Case I. Let $\delta>1$. Since S is not μ measure preserving, G can not be compact or discrete. Thus for any positive number ε there exists a nonvoid open set U which satisfies $\mu(U)<\varepsilon$. Now let V be a compact neighborhood of the identity e of G. Let $W=\bigcup_{n=0}^{\infty} S^{-n}(V)$. Then

$$\mu(W)\leqq\sum_{n=0}^{\infty}\mu(S^{-n}(V))=\left(\sum_{n=0}^{\infty}\delta^{-n}\right)\mu(V)=\frac{\delta}{\delta-1}\mu(V)<\infty.$$

Clearly $S(W)\supset W$, so we have $S^n(G\cap W^c)\cap W=\phi$ for $n=0,1,2,\cdots$. Since W is σ-compact, there exists a σ-compact open subgroup H of G such that $W\subset H$. Therefore there exists a Borel set E such that

$$E\subset G\cap W^c \quad \text{and} \quad 0<\mu(E)<\frac{\delta-1}{2\delta}\mu(V).$$

Then

$$\mu\left(\bigcup_{n=1}^{\infty} S^{-n}(E)\right)<\mu(V)/2.$$

Let $F=\bigcup_{n=-\infty}^{\infty} S^n(E)$. Then $S^{-1}(F)=F$, $\mu(F)>0$ and $\mu(G\cap F^c)>0$. So S is not ergodic with respect to μ.

Case II. Let $\delta<1$. Then $S^{-1}(x)=T^{-1}(a^{-1})\cdot T^{-1}(x)$ is not ergodic by Case I, whence S is not ergodic.

The proof is complete.

By Theorems 1 and 2, an affine transformation $S(x)=a\cdot T(x)$ of a locally compact group G which is ergodic with respect to a left Haar measure μ on G is *bi*-continuous and μ measure preserving. Thus an ergodic S induces a unitary operator $U(S)$ of $L^2(G,\mu)$ as follows

$$(U(S)f)(x)=f(S(x)) \quad \text{for} \quad f\in L^2(G,\mu).$$

Let for y in G, $V(y)$ be the unitary operator of $L^2(G,\mu)$ which is defined by

$$(V(y)f)(x)=f(yx) \quad \text{for} \quad f\in L^2(G,\mu).$$

The following two lemmas are contained in [5].

Lemma 1. *Let $S(x)=a\cdot T(x)$ be an affine transformation of a locally compact group G which is ergodic with respect to a left Haar measure μ on G. Then*

$$V(S^n(y))=[U(T)]^{-n}V(y)[U(S)]^n$$

for every integer n and every y in G.

Lemma 2. *Let H be a complex Hilbert space, let A be a bounded operator and U_1 and U_2 unitary operators on H. Then for given ξ and η in H, the sequence $\langle AU_1^n(\xi), U_2^n(\eta)\rangle_{n=-\infty}^{\infty}$ is a sequence of Fourier-Stieltjes coefficients of some complex regular measure on the torus $K=\{\exp(i\theta)\,|\,0\leqq\theta<2\pi\}$.*

Theorem 3. *If G is a locally compact totally disconnected non-discrete group which has an ergodic affine transformation $S(x)=a\cdot T(x)$ with respect to a left Haar measure μ on G then G is compact.*

Proof. Let N be a compact open subgroup of G and let μ be normalized so that $\mu(N)=1$. Let $U(S)$ and $V(y)$ be as above. For y in G and n an integer we define

$$\begin{aligned} a_n(y)&=\langle V(y)[U(S)]^n\chi_N, [U(T)]^n\chi_N\rangle \\ &=\langle [U(T)]^{-n}V(y)[U(S)]^n\chi_N, \chi_N\rangle, \end{aligned}$$

where χ_N is the indicator function of N. Then from Lemma 1 we observe

$$\begin{aligned} a_n(y) &= \langle V(S^n(y))\chi_N, \chi_N\rangle \\ &= \int_G \chi_N(S^n(y)x)\chi_N(x)\,d\mu(x) \\ &= \begin{cases} 1 & \text{if} \quad y \in S^{-n}(N) \\ 0 & \text{if} \quad y \notin S^{-n}(N). \end{cases} \end{aligned} \tag{1}$$

Thus

$$a_n(S(y)) = \langle V(S^{n+1}(y))\chi_N, \chi_N\rangle = a_{n+1}(y).$$

By Lemma 2 and (1), the sequence $\langle a_n(y)\rangle_{n=-\infty}^{\infty}$ is a sequence of Fourier-Stieltjes coefficients of some idempotent measure on the torus K. Hence the sequence $\langle a_n(y)\rangle_{n=-\infty}^{\infty}$ differs from a periodic sequence at most finitely many places (see for example [4, 3.1.6]). So the set $\mathfrak{M}$ of all sequences $\langle a_n\rangle_{n=-\infty}^{\infty}$ which is of the form $\langle a_n\rangle_{n=-\infty}^{\infty} = \langle a_n(y)\rangle_{n=-\infty}^{\infty}$ for some y in G is countable. For $\langle a_n\rangle_{n=-\infty}^{\infty} \in \mathfrak{M}$, let $M(\langle a_n\rangle_{n=-\infty}^{\infty})$ be the set defined by

$$\begin{aligned} M(\langle a_n\rangle_{n=-\infty}^{\infty}) &= \{y \in G \mid \langle a_n\rangle_{n=-\infty}^{\infty} = \langle a_n(y)\rangle_{n=-\infty}^{\infty}\} \\ &= \bigcap_{n=-\infty}^{\infty} S^{-n}(N^{\varepsilon_n}), \end{aligned}$$

where $N^{\varepsilon_n} = N$ if $\varepsilon_n = a_n = 1$ and $N^{\varepsilon_n} = G \cap N^c$ if $\varepsilon_n = a_n = -1$. Since N is open and closed, $M(\langle a_n\rangle_{n=-\infty}^{\infty})$ is an intersection of open and closed sets, so closed. By the Baire category theorem, there exists at least one sequence $\langle a_n\rangle_{n=-\infty}^{\infty}$ in $\mathfrak{M}$ such that $M(\langle a_n\rangle_{n=-\infty}^{\infty})$ has non-void interior. Then the set

$$\begin{aligned} M^*(\langle a_n\rangle_{n=-\infty}^{\infty}) &= \bigcup_{j=-\infty}^{\infty} S^j(M(\langle a_n\rangle_{n=-\infty}^{\infty})) \\ &= \{y \in G \mid \langle a_n\rangle_{n=-\infty}^{\infty} = \langle a_{n+k}(y)\rangle_{n=-\infty}^{\infty} \text{ for some integer } k\} \end{aligned}$$

must be almost all of G since S is ergodic.

Let $a_n = 0$ for all but finitely many n. Let $k = 1 + \max\{|m-n| \mid a_m = a_n = 1\}$. Since G is non-discrete, μ is not atomic. Thus for $\varepsilon = 1/(2k+1)$, there exists a neighborhood W of the identity e of G such that $\mu(W) < \varepsilon$ and $W \subset N$. Since $\{S^j(W) \mid j = 0, \pm 1, \pm 2, \cdots\}$ covers almost all of N and S is μ measure preserving, it follows that $\{j \mid S^j(W) \cap N \neq \phi\}$ contains at least $(2k+1)$ integers. So there exists an integer i such that $S^i(W) \cap N \neq \phi$ and $|i| \geqq k$. Hence

$$N \cap S^{-i}(N) \supset W \cap S^{-i}(N) \neq \phi$$

and

$$M^*(\langle a_n\rangle_{n=-\infty}^{\infty}) \cap (N \cap S^{-i}(N)) = \phi.$$

This is impossible, thus $a_n = 1$ for infinitely many n. For such an essentially periodic sequence there exists a positive integer p such that in every interval of length p there exists at least one integer n such that $a_n = 1$, and so

$$M^*(\langle a_n\rangle_{n=-\infty}^{\infty}) \subset N \cup S(N) \cup \cdots \cup S^p(N).$$

This establishes Theorem 3.

References

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