49. *Notes on Finite Left Amenable Semigroups**)

By Takayuki TAMURA
University of California, Davis, California, U. S. A.

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Let S be a semigroup and $B(S)$ be the Banach space of all bounded complex or real valued functions on S. A semigroup S is called left [right] amenable if there is on $B(S)$ a mean m, that is, a linear functional m for which $\|m\| = 1$ and $m(x) \geq 0$ if $x \geq 0$ on S and which is invariant under left [right] translations of elements of $B(S)$ by elements of S, in other words, $m(\alpha f) = m(f)$ where $(\alpha f)(x) = f(\alpha x)$, $f \in B(S)$, $x \in S$, α complex or real numbers, S is called amenable if S is left amenable and right amenable.

In (3I′), at p. 11 of [2] we can see the following proposition due to Rosen [5]:

Proposition 1. *A finite semigroup S is left amenable if and only if it has a unique minimal right ideal R. Then this right ideal is the union of the disjoint minimal left ideal $L_1, \cdots, L_k$ of S; each left ideal is a group, and all these groups are isomorphic. If u_i is the identity element of the group L_i, then $u_i u_j = u_j$ for all $i, j \leq k$, and if U is the set of these u_i, $R = L_i \times U$, and the left invariant means on S are supported on R and are exactly averaging over L_i crossed with arbitrary means on U.*

The statement concerning the minimal right ideal means that the right ideal is a right group [1], i.e. the direct product of a group and a right zero semigroup. Furthermore it is the kernel i.e. the minimal ideal. In this paper the author notices that a finite left amenable semigroup is characterized by left zero indecomposability of ideals.

By a left zero semigroup we mean a semigroup satisfying the identity $xy = x$. Every semigroup S has a smallest left zero congruence ρ_0, that is, ρ_0 is a congruence such that S/ρ_0 is a left zero semigroup, and ρ_0 is contained in all congruences ρ such that S/ρ are left zero semigroups. If ρ_0 is the universal relation, $\rho_0 = S \times S$, then S is called left zero indecomposable. Refer undefined terminology to [1].

Theorem 2. *Let S be a finite semigroup. The following are equivalent:*

(1) *Every ideal of S is left zero indecomposable.*
(2) *The kernel K of S is a right group, $|K| \geq 1$.*
(3) *S has a unique minimal right ideal.*

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To prove the theorem, we need a lemma. Let D be a completely simple semigroup and let $D=\mathcal{M}(G;\Delta,I;P)$ be its Rees regular matrix representation, G a group, P a sandwich (I,Δ)-matrix over G. In other words

$$D=\{(x;\lambda,\mu);x\in G,\lambda\in\Delta,\mu\in I\}$$

and the operation is defined by

$$(x;\lambda,\mu)(y;\xi,\eta)=(xp_{\mu\xi}y;\lambda,\eta)$$

where $p_{\mu\xi}$ is the (μ,ξ)-element of P.

Lemma 3. *Let f be a transformation of a set I i.e. a map of I into I. Let g be a map of I into a group G. Let $S=\mathcal{M}(G;\Delta,I;P)$. For a pair (g,f), a transformation $\varphi_{g,f}$ of S is defined as follows:*

$$(x;\lambda,\mu)\varphi_{g,f}=(x\cdot(\mu g);\lambda,\mu f).$$

Then $\varphi_{g,f}$ is a right translation of S. Every right translation is determined by g and f in this manner.

Proof. Let φ be a right translation of S, [1], and let $(x;\lambda,\mu)\varphi=(x';\lambda',\mu')$. Since $(x;\lambda,\mu)=(p_{\nu\lambda}^{-1};\lambda,\nu)(x;\lambda,\mu)$, applying φ to the both sides, we have $\lambda'=\lambda$. Next we will prove that μ' is independent of x and λ; and x' is independent of λ.

Let

$$(x;\lambda_1,\mu)\varphi=(x';\lambda_1,\mu'),$$
$$(x;\lambda_2,\mu)\varphi=(x'';\lambda_2,\mu'').$$

Applying φ to

$$(x;\lambda_1,\mu)=(p_{\eta\lambda_2}^{-1};\lambda_1,\eta)(x;\lambda_2,\mu),$$

we have

$$(x';\lambda_1,\mu')=(x'';\lambda_1,\mu'')$$

hence

$$x'=x'',\ \mu'=\mu''.$$

Thus we see that φ induces a transformation of I, $\mu\to\mu'$, denoted by f and the map $x\to x'$, is independent of λ. Let $\bar{g}$ denote the maps $x\to x'$. Applying φ to $(x;\lambda,\mu)(y;\lambda,\mu)$, we have $(xp_{\mu\lambda}y)\bar{g}=xp_{\mu\lambda}(y\bar{g})$. Let $z=xp_{\mu\lambda}$. Then $(zy)\bar{g}=z(y\bar{g})$ for all $z,y\in G$. However, we know a right translation of a group is inner, that is, there is an element a of G such that $x\bar{g}=xa$ in which a depends on μ. We denote it by $a=\mu\bar{g}$. The proof of the converse is routine.

Similarly we can prove:

Lemma 3′. *For a pair of a transformation f of a set Δ and a map g of Δ into G, a left translation $\psi_{g,f}$ of S is given by*

$$(x;\lambda,\mu)\psi_{g,f}=(\lambda g\cdot x;\lambda g,\mu).$$

Proof of Theorem 2.

(1)→(2). Since S is finite S has a kernel K which is finite simple, hence completely simple. Let $K=\mathcal{M}(G;\Delta,I;P)$. Define a relation ρ on K by

$$(x;\alpha,\beta)\rho(y;\gamma,\delta)\qquad\text{iff }\alpha=\gamma.$$

It is easy to see that ρ is a left zero congruence on K. In order that K be left zero indecomposable, Δ has to be trivial, $|\Delta|=1$. Now $K=\{(x;1,\mu): x\in G, \mu\in I\}$ and K is isomorphic to the direct product $G\times I$ of a group G and a right zero semigroup I under a map, $(x;1,\mu)\to(xp_{\mu 1},\mu)$.

(2)→(3). Let K be the kernel of S and assume that K is a right group. Let I be a minimal right ideal of S. Let $a\in I, b\in K$. Then $ab\in K$, and

$$K=abK\subseteq aS\subseteq I.$$

By minimality of I, we have $I=K$. This shows that a minimal right ideal of S is unique.

(3)→(1). Let $K=\mathcal{M}(G;\Delta,I;P)$ be the kernel of S. Suppose $|\Delta|>1$. Let $\lambda_1,\lambda_2\in\Delta, \lambda_1\neq\lambda_2$. Let $(x;\lambda_1,\mu_1), (y;\lambda_2,\mu_2)\in K$. Recalling that ideal extensions of K are given by translations of K and using Lemma 3, we have

$$(x;\lambda_1,\mu_1)S\cap(y;\lambda_2,\mu_2)S=\emptyset,\ \lambda_1\neq\lambda_2.$$

This is a contradiction to the assumption because the two right ideals would have to contain the unique minimal right ideal in their intersection. As shown in the proof of (1)→(2), K is a right group. Let I be an arbitrary ideal of S. $K\subseteq I\subseteq S$. I is the ideal extension of K by Z where Z is a semigroup with zero, that is, $I/K=Z$. We will prove here that I is left zero indecomposable. Let ρ be a left zero congruence on I. Let a, b be arbitrary elements of I and $c\in K$. Let e be a left identity element of K. Then since ρ satisfies $xy\,\rho\,x$ for all $x, y\in I$, we have

$$a\,\rho\,ac\,\rho\,eac\,\rho\,e\,\rho\,ebc\,\rho\,bc\,\rho\,b.$$

Thus ρ is the universal relation i.e. $I\times I$, that is, I is left zero indecomposable.

Remark. The condition (1) is also equivalent to

(1′) The kernel of S is left zero indecomposable.

If finiteness is not assumed, (1) is not equivalent to left amenability. For example a free group on two generators is not amenable. (See [2].) (1) cannot be replaced by the following:

(1″) S is left zero indecomposable.

For example let $S=\{a,b,c,d\}$ be a semigroup defined by

$xy=x$ for $x=a$ or b and for all $y\in S$.

$cy=y$ for all $y\in S$

$da=b;\ db=a,\ dc=d,\ d^2=c.$

S is left zero indecomposable but the kernel $\{a,b\}$ is a left zero semigroup.

Combining Theorem 2 with its dual case we have

Theorem 3. *Let S be a finite semigroup. The following are* *equivalent*:

(4) *Every ideal of S is rectangular band indecomposable.*

(5) *The kernel K of S is a group,* $|K| \geq 1$.

(6) *S has a unique minimal right ideal and a unique minimal left ideal.*

We have proved that a finite left amenable semigroup S is the ideal extension of a finite right group K by a finite semigroup W with zero. Since K is weakly reductive, the method of Theorem 4.20 or Theorem 4.21 in [1] can be applied to the construction of S i.e. the ideal extension of K by W. Though we will not describe the ideal extension here, we would like to notice something about the translation semigroup and the translational hull of K.

Let $K = G \times R$, G a group, R a right zero semigroup. Let Φ be the semigroup of all ordered pairs $((g, f))$ of $g : R \rightarrow G$ and $f : R \rightarrow G$. The operation in Φ is defined by

$$((g_1, f_1)) \cdot ((g_2, f_2)) = ((g_1 * f_1 g_2, f_1 f_2))$$

where

$$(\alpha)(g_1 * f_1 g_2) = (\alpha g_1)(\alpha f_1 g_2), \qquad (\alpha)(f_1 g_2) = (\alpha f_1) g_2$$
$$(\alpha)(f_1 f_2) = (\alpha f_1) f_2, \qquad \alpha \in R.$$

By Lemma 3 we see that Φ is isomorphic to the semigroup of all right translations of K. The semigroup of all left translations ψ_a of K is isomorphic to $G : \psi_a \rightarrow a, a \in G$. It can be proved that $\varphi_{g,f}$ is linked with ψ_a if and only if g is a zero map, i.e. $\alpha g = a$ for all $\alpha \in R$; therefore the translational hull $\mathscr{H}$ of K is isomorphic to the direct product of G and the full transformation semigroup $\mathscr{T}_R$ over the set R.

$$\mathscr{H} \cong G \times \mathscr{T}_R = \{(a, f) : a \in G, f \in \mathscr{T}_R\}.$$

The inner part of $\mathscr{H}$ i.e. the subsemigroup of $\mathscr{H}$ which is identified with K is isomorphic to $G \times \bar{R}$, $\bar{R}$ is the semigroup of all zero maps of R into R.

Recently Melven Krom and Myren Krom have obtained in [4] a necessary and sufficient condition for a semigroup to be left amenable in terms of subsets of the semigroup. As its consequence they have had

Theorem 4. *A finite semigroup S is left amenable if and only if there is a nonempty subset Q of S such that*

$$\cap \{aQ : a \in S\} = Q$$

Consequently the existence of such a Q is equivalent to one of (1), (2) *and* (3) *of Theorem* 2.

Remark to Theorem 2. This paper has not been explicitly concerned with the "left [right] amenability". In Theorem 2, however, if we put the condition:

(0) *S is left amenable.*

Then the implications (0)→(1) and (3)→(0) are easily obtained as follows:

(0)→(1) If S is left amenable, every right ideal of S, hence every ideal of S, is left amenable. This is due to Frey [3] (See (3L′) in [2]). Suppose an ideal I is homomorphic to a non-trivial left zero semigroup L. L is also left amenable by (3C) in [2]. This is a contradiction because a left zero semigroup is not left amenable by the remark after (3I) in [2]. Therefore I has to be left zero indecomposable.

(3)→(0). This is given by Rosen [5] (See (3I′) in [2]).

Problem. The algebraic structure of finite left amenable semigroups has been clarified. How is the algebraic theory related to the left translation invariant mean m? Let S be an ideal extension of $G\times R$ by W. If m_1, m_2, m_3 are means of G, R and W respectively, what relationship is there between these means and a mean on S?

References

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