48. *Abelian Groups and $\mathfrak{N}$-Semigroups**)

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§1. **Introduction.** A commutative archimedean cancellative semigroup without idempotent is called an $\mathfrak{N}$-semigroup. The author obtained the following [3 or 1, p. 136].

Theorem 1. *Let K be an abelian group and N be the set of all non-negative integers. Let I be a function $K\times K\rightarrow N$ which satisfies the following conditions:*

(1) $I(\alpha,\beta)=I(\beta,\alpha)$ *for all* $\alpha,\beta\in K$.

(2) $I(\alpha,\beta)+I(\alpha\beta,\gamma)=I(\alpha,\beta\gamma)+I(\beta,\gamma)$ *for all* $\alpha,\beta,\gamma\in K$.

(3) $I(\varepsilon,\varepsilon)=1$. ε *being the identity element of* K.

(4) *For every* $\alpha\in K$ *there is a positive integer* m *such that* $I(\alpha^m,\alpha)>0$.

We define an operation on the set $S=N\times K=\{(m,\alpha): m\in N, \alpha\in K\}$ *by* $(m,\alpha)(n,\beta)=(m+n+I(\alpha,\beta),\alpha\beta)$.

Then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup is obtained in this manner. S is denoted by $S=(K;I)$.

To prove the theorem in [3] we used the fact $\bigcap_{n=1}^{\infty} a^nS=\emptyset$, and a group-congruence was defined, which are still effective in the case where cancellation is not assumed. However the quotient group of an $\mathfrak{N}$-semigroup gives another proof of the latter half of the theorem. In this paper we study the relationship between an $\mathfrak{N}$-semigroup and abelian group as the quotient group. Theorm 2 states the relationship between the I-function of an $\mathfrak{N}$-semigroup and the factor system of the quotient group as the extension. Theorem 4 is the main theorem of this paper, which asserts the existence of maximal $\mathfrak{N}$-subsemigroups of a given abelian group. Theorem 5 is an application of Theorem 4 to the extension theory of abelian groups.

§2. **Proof of a part of the latter half of Theorem 1.** Let S be an $\mathfrak{N}$-semigroup and G be the quotient group of S, i.e. the smallest group into which S can be embedded. We may assume $S\subset G$. Let $a\in S$. The element a is of infinite order in G. Let A be the infinite cyclic group generated by a. Let $G_a=G/A$. G_a is called the structure group of S with respect to a. G is the disjoint union of the congruence classes of G modulo A.

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$$G=\bigcup_{\xi\in G_a} A_\xi$$

where A_ξ is the class corresponding to $\xi \in G_a$ and in particular $A_\varepsilon = A$. we will prove

(5) $$S\cap A_\xi \neq \emptyset \quad \text{for all } \xi \in G_a.$$

Let $x\in A_\xi$. Recalling the definition of the quotient group, $x=bc^{-1}$ for some $b, c\in S$, or $xc=b$. Since S is archimedean, $cd=a^m$ for some $d\in S$ and some positive integer m. Then $xc=b$ implies $xa^m=bd$, that is, $xa^m \in S\cap A_\xi$. Let $S_\xi=S\cap A_\xi$ for $\xi\in G_a$. In particular $S_\varepsilon = S\cap A_\varepsilon = \{a^i : i=1,2,3,\cdots\}$. Then $S=\bigcup_{\xi\in G_a} S_\xi$, and S is homomorphic onto G_a under the restriction of the natural mapping, $G\to G_a$, to S. Let $x_\xi \in A_\xi$ and let $\{x_\xi : \xi \in G_a\}$ be a complete representative system of G modulo A. We will prove there is an integer $\delta(\xi)$ such that for each ξ,

$$\text{if } \delta(\xi)\leq m,\ x_\xi a^m \in S \text{ but if } m<\delta(\xi),\ x_\xi a^m \notin S.$$

It is obvious that $x_\xi a^i \in S_\xi$ implies $x_\xi a^j \in S_\xi$ for all $j\geq i$. Since certainly $S_\xi \neq \emptyset$ by (5), it is sufficient to show $S_\xi \neq \{x_\xi a^i : i=0, \pm 1, \pm 2, \cdots\}$. Suppose the contrary. Then $x_\xi \in S_\xi$. By archimedeaness $yx_\xi = a^k$ for some $y\in S$ and some $k>0$. Then $yx_\xi a^i = a^{k+i} \in S$ for all integers i; hence $A\subset S$. This is a contradiction to $S_\varepsilon = \{a^i : i\geq 1\}$. Let $p_\xi = a^{\delta(\xi)} x_\xi$. p_ξ cannot be divisible by a, in S. S_ξ contains p_ξ; in particular $p_\varepsilon = a$. p_ξ is called a prime with respect to a. Therefore for any element x of S there is $m\geq 0$ and p_ξ such that $x=a^m p_\xi$ where m, p_ξ are unique and if x itself is a prime, $m=0$. For the remaining part the same argument as in the original paper is effective.

Theorem 2. *Let S be an $\mathfrak{R}$-semigroup, $S=(K; I)$. The quotient group G of S is the abelian extension of the additive group Z of all integers by the abelian group K with respect to the factor system $c(\alpha, \beta)$*[1] *defined by*

$$c(\alpha,\beta)=I(\alpha,\beta)-1.$$

Proof. Let $G=\{((x,\alpha)) : x\in Z, \alpha\in K\}$ in which

$$((x,\alpha))((y,\beta))=((x+y+c(\alpha,\beta), \alpha\beta)),\quad c(\alpha,\beta)=I(\alpha,\beta)-1$$

and

$$((x,\alpha))^{-1}=((-x-c(\alpha,\alpha^{-1}), \alpha^{-1}))$$

G is the extension of $A=\{((x,\varepsilon)) ; x\in Z\}$ by K. Now let $S'=\{((n,\alpha)) ; \alpha\in K, n=1,2,\cdots\}$. Then $S\cong S'$ by the map $f: (n,\alpha)\to((n+1,\alpha))$. Next we will prove G is generated by S' in the sense of groups:

$$((0,\alpha))=((n,\alpha))((n,\varepsilon))^{-1}$$

and if $x>0$,

$$((-x,\alpha))=((n,\alpha))((n+x,\varepsilon))^{-1}.$$

The element $(0,\varepsilon)$ of S is mapped to $((1,\varepsilon))$ of S'. It is easy to see that the structure group of S' with respect to $((1,\varepsilon))$ is isomorphic to K.

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1) See [2]. c satisfies $c(\alpha,\beta)=c(\beta,\alpha)$, $c(\alpha,\beta)+c(\alpha\beta,\gamma)=c(\alpha,\beta\gamma)+c(\beta,\gamma)$, $c(\varepsilon,\varepsilon)=0$ (ε identity).

§ 3. Maximal $\mathfrak{N}$-subsemigroup. Let G be an abelian non-torsion group, S be an $\mathfrak{N}$-subsemigroup of G and A be an infinite cyclic subgroup $[a]$ of G generated by an element a of S.

Lemma 3. *The following are equivalent:*

(6) *G is the quotient group of S.*

(7) *$G=A\cdot S$.*

(8) *S intersects each congruence class of G modulo A.*

Proof. (8) immediately follows from (6) as a consequence of (5) in § 2. (7)→(6), (8)→(7) are obvious.

Theorem 4. *Let G be an abelian group which is not torsion. Let a be an element of infinite order of G. There exists a (maximal) $\mathfrak{N}$-subsemigroup S containing a such that G is the quotient group of S.*

Proof. The operation in G and S will be denoted by $+$. Let D be an abelian divisible group into which G can be embedded. According to the theory of abelian groups, D is the direct sum (i.e. the restricted direct product): [See, for example, 2].

$$D=\sum_{\lambda\in\Lambda} R_\lambda \oplus \sum_{\mu\in M} C_\mu$$

where R_λ's are the copies of the group of all rational numbers under addition and C_μ's are the quasicyclic groups $\bigcup_{n=1}^{\infty} C(p_\mu^n)$, p_μ's being various primes. Let $\pi_\lambda: D\to R_\lambda$ and $\pi_\mu: D\to C_\mu$ be the projections of D to R_λ and C_μ respectively. Since a is of infinite order there is $\lambda_1\in\Lambda$ such that $\pi_{\lambda_1}(a)\neq 0$. The reason is this: Suppose $\pi_\lambda(a)=0$ for all $\lambda\in\Lambda$. Then only a finite number of the components $\pi_\mu(a)$, $\mu\in M$, are not 0, therefore a would be of finite order. We assume $\pi_{\lambda_1}(a)>0$.

Define $$S^*=\{x\in D: \pi_{\lambda_1}(x)>0\}.$$

It is obvious that S^* is commutative, cancellative and has no idempotent, and $a\in S^*$. To prove S^* is archimedean, we see first

$$S^*\cong P_{\lambda_1}\oplus\Big(\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq\lambda_1}} R_\lambda\oplus\sum_{\mu\in M} C_\mu\Big)$$

where $a\in P_{\lambda_1}$ and P_{λ_1} is the semigroup of all positive rational numbers under addition. P_{λ_1} is archimedean, and the second factor (i.e. the sum of all factors within the parentheses) is a group, hence archimedean; therefore it is easy to show that S^* is archimedean. Thus it has been proved that S^* is an $\mathfrak{N}$-semigroup containing a. We will prove $D=A+S^*$. Let $x\in D$. Choose a positive integer n such that $n\pi_{\lambda_1}(a)+\pi_{\lambda_1}(x)>0$. Then $n\cdot a+x\in S^*$, hence $D\subseteq A+S^*$. The other direction is obvious. Next let $S=G\cap S^*$. S will be one of $\mathfrak{N}$-semigroups which are claimed. Clearly S contains a and S is commutative, cancellative and has no idempotent. Since S^* intersects all the congruence classes of D modulo A (by Lemma 3), S^*, hence S does all the congruence classes of G modulo A, that is, $G=A+S$. It remains to show that S is archimedean. Let $x, y \in S$. Since S^* is archimedean there are a positive integer m and an element $z \in S^*$ such that $m \cdot x = y + z$. On the other hand since $x, y \in G, z \in G$. Consequently $z \in S$. It goes without saying that G is the quotient group of S by Lemma 3. To prove the existence of a maximal one, use Zorn's lemma: We can easily prove that if $S_\xi, \xi \in \Xi$, are $\mathfrak{N}$-semigroups containing a as above and if $S_\xi \subset S_\eta$ for $\xi < \eta$, then $\bigcup_{\xi \in \Xi} S_\xi$ is also such one.

§ 4. Application to abelian group theory.

Theorem 5. *Let K be an abelian group and A be the group of all integers under addition. If G is an abelian extension of A by K with respect to a factor system $f(\alpha, \beta), K \times K \to A$, then there exists a factor system $g(\alpha, \beta)$ such that*

(9) $g(\alpha, \beta) \geqq 0$

(10) *$g(\alpha, \beta)$ is equivalent*[2] *to $f(\alpha, \beta)$.*

Proof. By the assumption, let $G = \{((m, \alpha)) : \alpha \in K, m = 0, \pm 1, \pm 2, \cdots\}$ in which

(11) $$((m, \alpha))((n, \beta)) = ((m + n + f(\alpha, \beta), \alpha\beta)).$$

Let ε be the identity element of K. By Theorem 4 there is an $\mathfrak{N}$-subsemigroup S containing $((1, \varepsilon))$ such that G is the quotient group of S. Recalling the proof in § 1, for each $\alpha \in K$ there is an integer $\delta(\alpha)$, in particular $\delta(\varepsilon) = 1$, such that $((m, \alpha)) \in S$ for all $m \geqq \delta(\alpha)$. Hence

(12) $$S = \{((m, \alpha)) : m \geq \delta(\alpha), \alpha \in K\}.$$

Since S is closed with respect to the group operation,

$$m + n + f(\alpha, \beta) \geqq \delta(\alpha\beta)$$

holds for all $m \geqq \delta(\alpha)$ and all $n \geqq \delta(\beta)$. This is equivalent to

$$\delta(\alpha) + \delta(\beta) + f(\alpha, \beta) \geq \delta(\alpha\beta).$$

Let $g(\alpha, \beta) = f(\alpha, \beta) + \delta(\alpha) + \delta(\beta) - \delta(\alpha\beta)$. Then $g(\alpha, \beta)$ is a factor system which is equivalent to $f(\alpha, \beta)$ and

$$g(\alpha, \beta) \geqq 0.$$

Problem. Can Theorem 5 be directly proved without using Theorem 4? Can Theorem 5 be generalized to the case where A is an ordered group?

Addendum. Let G be a non-torsion abelian group, that is, the abelian extension of the additive group A of all integers by an abelian group K with respect to a factor system f. If δ is a map, $K \to A$, satisfying

(i) $\delta(\varepsilon) = 1$ ε being the identity of K,

(ii) $f(\alpha, \beta) + \delta(\alpha) + \delta(\beta) - \delta(\alpha\beta) \geqq 0$ for all $\alpha, \beta \in K$,

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2) See [2]. $g(\alpha, \beta)$ is said to be equivalent to $f(\alpha, \beta)$ if there is $\varphi: K \to A$ such that $g(\alpha, \beta) = f(\alpha, \beta) + \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)$.

(iii) for every $\alpha \in K$ there is a positive integer m such that $f(\alpha, \alpha^m)+\delta(\alpha)+\delta(\alpha^m)-\delta(\alpha^{m+1})>0$,

and if S is defined by (12) with (11), then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup containing $((1, \varepsilon))$ whose quotient group is G can be obtained in this manner.

References

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