# 82. Notes on Modules. III

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In this paper we discuss the Kertész' radical for modules, and among other we show that this radical fails to be a ring radical in the sense of Amitsur and Kurosh. We refer yet concerning this topic to our earlier papers [6], [7].

Following Kertész [3], for an arbitrary ring $A$ and for any right $A$-module $M$, we consider the set

(1) $$K(M)=\{X_{/}X\in M, \quad XA\subseteq \Phi(M)\}$$

where $\Phi(M)$ denotes the Frattini $A$-submodule of $M$. (That is, $\Phi(M)$ is the intersection of all maximal submodules of $M$, and $\Phi(M)=M$ for modules $M$ having no maximal $A$-submodules.) Obviously, $K(M)$ is an $A$-submodule of $M$. Calling an $A$-submodule $N$ of $M$ homoperfect, if

(2) $$MA+N=M$$

holds, then (1) implies by Kertész [3], that $K(M)$ coincides with the intersection of all homoperfect maximal $A$-submodules of $M$

**Example.** For a prime number $p$ let $A$ be the ring generated by the $3\times 3$ matrices over the field of $p$ elements:

(3) $$x=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\qquad y=\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix}$$

Then $A$ is a noncommutative ring with $p^2$ elements and with the multiplication:

(4)

| | $x$ | $y$ |
|---|---|---|
| $x$ | 0 | $x$ |
| $y$ | 0 | $y$ |

By a routine calculation it can be verified that the principal right ideal $(y)_r$ of $A$ is a homoperfect maximal right ideal, but $(y)_r$ is neither modular, nor quasimodular in $A$.

Furthermore, for the Kertész radical $K_r(A)$ of the $A$-right module $A$, one has by

(5) $$(x)_r\cap(y)_r=0$$

obviously $K_r(A)=0$, being also $(x)_r$ homoperfect and maximal in $A$. The Jacobson radical $F(A)$ of $A$ now coincides with $(x)_l=K_l(A)$, denoting $K_l(A)$ the left-right dual of $K_r(A)$

Therefore, this ring $A$ has the property, that

(6) $$0=K_r(A)\neq K_l(A)=F(A)$$

**Remark 1.** For an antiisomorphic image $A'$ of the ring $A$ of the above example evidently holds

$$0=K_l(A')\neq K_r(A')=F(A') \quad (7)$$

**Theorem 1.** *For an arbitrary cardinality $\mathfrak{M}$ there exists a ring $A$ with $\mathfrak{M}$ different elements and with conditions $0=K_r(A)\neq K_l(A)=F(A)$ if and only if $\mathfrak{M}$ is not a quadratfree finite number.*

**Proof.** If $\mathfrak{M}$ is a quadratfree finite number, and $A$ has exactly $\mathfrak{M}$ different elements, then $A$ is a ringdirect sum of rings of prime order. These components are commutative rings, therefore also $A$ is commutative, consequently $K_r(A)=F(A)$.

But in the case, when $\mathfrak{M}$ is finite and not quadratfree, then $\mathfrak{M}=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ with $\alpha_i\geqq 2$ at least for an $i$, with different prime numbers $p_j$. Assume that $i=1$ and $p_1=p$. Let our ring $B$ be the ringdirect sum of the ring $A$ from the above example, of $(\alpha_1-2)$ copies of fields of order $p$ and of $\alpha_j$ copies of fields of order $p_j$ for every $p_j\neq p$. Then one has obviously $|B|=\mathfrak{M}$ and $0=K_r(B)\neq K_l(B)=F(B)$.

Thirdly, if $\mathfrak{M}$ is an infinite cardinality, then let $C$ be the ringdirect sum of the ring $A$ from the example and of a field with $\mathfrak{M}$ elements. This field can be taken, as a field extension of the rational number field with the transcendence grad $\mathfrak{M}$. Then evidently $|C|=\mathfrak{M}$ and

$$0=K_r(C)\neq K_l(C)=F(C), \quad (8)$$

which completes the proof of Theorem 1.

**Remark 2.** The above ring $C$, constructed for an infinite $\mathfrak{M}$ as a right $C$-module $C$, is completely reducible, without nonzero left annihilators, but with the nonzero right annihilator $(x)_r=F(C)$. A right completely reducible ring $A$ has no nonzero right annihilators if and only if $C$ is semisimple in the sense of Jacobson, and $C$ satisfies the minimum condition for principal right ideals. (Cf. F. Szász [7].)

**Remark 3.** By the present author [8] was proved the existence of a right having a quasimodular maximal, but not modular right ideal. Calling an ideal $Q$ of a ring $A$ quasiprimitive, if there exists a quasimodular maximal right ideal $R$ of $A$ satisfying $Q=\{x\,;\,x\in A, Ax\subseteq R\}$, the equivalence of primitive and quasiprimitive ideals can be verified (cf. Steinfeld [5], and in a sharper form F. Szász [9]). But, for a maximal right ideal of a ring "homoperfect", "quasimodular" and "modular" are three different concepts.

**Theorem 2.** *The twosided ideals $K_r$ and $K_l$ (Kertész radicals) satisfy $AK_r\subseteq\Phi_r\subseteq K_r\subseteq F$ and $K_lA\subseteq\Phi_l\subseteq K_l\subseteq F$ for any ring $A$, furthermore $K_r$ and $K_l$ are not radicals in the sense of Amitsur and Kurosh.*

**Proof.** By the definition (1) it is sufficient to verify only the last statements (cf. yet F. Szász [8]).

Assume that $K_r$ is a radical in the sense of Amitsur and Kurosh.

Then by Theorem 47 of Divinsky's book [1], any twosided ideal of a semisimple ring is also semisimple. But the ring $A$ of the earlier example of the present paper satisfies $K_r(A)=0$ with $K_r(F(A))=F(A)\neq 0$ for the Jacobson radical of $A$.

This completes the proof of Theorem 2.

**Theorem 3.** *For any ring $A$ the following conditions are equivalent:*

a) *$A$ is a semisimple Artin ring,*

b) *$A$ is a ring with twosided unity satisfying the minimum condition on principal right ideals and yet with the condition that $K(M)\cdot A=0$ for the Kertész $K(M)$ radical of every right $A$-module $M$ holds.*

**Proof.** a) implies b). By assumption a) follows, that is also a ring with twosided unity and with minimum condition on principal right ideals. Furthermore, any $A$-right module $M$ can be decomposed into a form

$$M=M_0\oplus M_1 \tag{9}$$

where $\oplus$ is a module direct sum, $M_0A=0$ and $M_1$ is an unitary $A$-module. This can be proved by Peirce decompositions. Moreover $M_1$ is a completely reducible $A$-right module, which implies $K(M_1)=0$ and $K(M)=M_0$ whence

$$K(M)\cdot A=0$$

Conversely, also b) implies a). Let $A$ be a ring having twosided unity, satisfying the minimum condition on principal right ideals and with $K(M)\cdot A=0$ for every right $A$-module $M$. Then $K_r(A)$ coincides with the Jacobson radical $F$ of $A$, and $FA=0$ implies by $1\in A$ evidently $F(A)=0$. Therefore, the right $A$-module $A$ is completely reducible by the author's paper [7]. Consequently $A$ is by $1\in A$ a semisimple Artin ring.

This completes the proof of Theorem 3.

## References

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