79. *On the Existence of a Potential Theoretic Measure with Infinite Norm*

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Introduction. Let R^m be the m-dimensional Euclidian space and $\phi(x,y)$ a lower semi-continuous function from $R^m \times R^m$ into $[0, +\infty]$. The ϕ-potential of a positive Radon measure μ in R^m is defined by

$$\phi\mu(x)=\int\phi(x,y)d\mu(y).$$

In the case that there exists at least such a positive measure ν that the support $S\nu$ is compact and the potential $\phi\nu(x)$ is continuous in the whole space R^m, we can consider the following classes of measures;

$$\mathcal{F}(\phi)=\{\nu\,;\,\nu\geqq 0,\ S\nu \text{ compact and } \phi\nu(x) \text{ continuous in } R^m\},$$

$$\mathcal{G}(\phi)=\left\{\mu\,;\,\mu\geqq 0 \text{ and } \int\phi\mu d\nu<+\infty \text{ for any } \nu\in\mathcal{F}(\phi)\right\}.$$

The aim of this paper is to answer affirmatively for a question posed by G. Anger [1]: Let $\phi_N(x,y)$ be the Newtonian kernel defined in R^m $(m\geqq 3)$. Is there a measure $\mu\in\mathcal{G}(\phi_N)$ with infinite norm? Moreover we study the same problem in case of α-kernel $\phi_\alpha(x,y)$.

1. Existence of a measure $\mu\in\mathcal{G}(\phi_N)$ with infinite norm.

The Newtonian kernel $\phi_N(x,y)$ in R^m $(m\geqq 3)$ is defined by

$$\phi_N(x,y)=|x-y|^{2-m},$$

where $|x-y|$ denotes the distance between two points x and y in R^m. Let $B_{a,r}$ be the closed ball with the center a and the radius r and $S_{a,r}$ the surface of the ball $B_{a,r}$. We introduce the class of measures

$$\mathcal{S}=\{\lambda\,;\ \text{spherical distribution with uniform density}\}.$$

Especially the spherical distribution with uniform density on $S_{a,r}$ is denoted by $\lambda_{a,r}$. It is well known that $\mathcal{S}$ is a non empty subset of $\mathcal{F}(\phi_N)$. Let us recall following potential theoretic principles,

Maximum principle: *If it holds that, for a constant V, $\phi\nu(x)\leqq V$ on the support $S\nu$ of a positive measure ν, then we have the same inequality in the whole space.*

Domination principle: *If it holds that, for a positive measure ν and an energy finite positive measure μ, $\phi\mu(x)\leqq\phi\nu(x)$ on the support $S\mu$, then we have the same inequality in the whole space.*

Lemma 1. *For a given positive measure μ, the mutual energy $\int\phi_N\mu d\nu$ is finite for any $\nu\in\mathcal{F}(\phi_N)$ if $\int\phi_N\mu d\lambda$ is finite for any $\lambda\in\mathcal{S}$.*

Proof. It is sufficient to show that, for any $\nu \in \mathcal{F}(\phi_N)$, we can choose such a suitable measure $\lambda \in \mathcal{S}$ that $\phi_N\lambda(x) \geqq \phi_N\nu(x)$ in the whole space R^m. The Newtonian kernel satisfying the maximum principle, the potential $\phi_N\nu(x)$ of a measure ν attains the maximum on the compact support $S\nu$. Let $\lambda_{a,r}$ be the spherical distribution with uniform density and the total mass $M>0$. It is well known that

$$(*)\qquad \phi_N\lambda_{a,r}(x)=\begin{cases}\dfrac{M}{r^{m-2}} & \text{in } B_{a,r}\\[2mm] \dfrac{M}{|x-a|^{m-2}} & \text{otherwise.}\end{cases}$$

Consequently, choosing a suitable center a, a radius r and a sufficiently large total mass M, we can pick up such a measure $\lambda_{a,r}$ that the corresponding ball $B_{a,r}$ contains the compact support $S\nu$ and $\phi_N\lambda_{a,r}(x) \geqq \phi_N\nu(x)$ on $S\nu$. The Newtonian kernel satisfying the domination principle, it follows that

$$\phi_N\mu_{a,r}(x) \geqq \phi_N\nu(x) \quad \text{in the whole space } R^m.$$

Lemma 2. *Let μ_s be the measure*

$$\mu_s=\sum_{n=1}^{+\infty} n^{m-s}\mu_n \quad \textit{for any real number } s\ (3<s\leqq 1+m),$$

where μ_n denotes a unit point mass on the sphere $S_{0,n}$ with the center the origin 0 and the radius n. Then μ_s is a positive Radon measure with infinite norm and we have $\int \phi_N\mu d\lambda<+\infty$ for any $\lambda \in \mathcal{S}$.

Proof. It is obvious that μ_s is a positive Radon measure and, owing to $s-m\leqq 1$, we have

$$\|\mu_s\|=\mu_s(R^m)=\sum_{n=1}^{+\infty} n^{m-s}=+\infty \quad \text{for any } s.$$

On account of (*), we have, for a measure $\lambda_{a,r}$ with the norm M,

$$\begin{aligned}\int\phi_N\mu_s d\lambda_{a,r}&=\int\phi_N\lambda_{a,r}d\mu_s\\ &=\int_{B_{a,r}}\phi_N\lambda_{a,r}d\mu_s+\int_{R^m-B_{a,r}}\phi_N\lambda_{a,r}d\mu_s\\ &=\sum_{|n-a|\leqq r}\frac{M}{r^{m-2}}\cdot\frac{1}{n^{s-m}}\\ &\quad+\sum_{|n-a|>r}\frac{M}{|n-a|^{m-2}}\cdot\frac{1}{n^{s-m}}\\ &<+\infty,\end{aligned}$$

because the first summation of the right hand side is obviously finite and the second summation is the same order of $\sum_{n=1}^{+\infty} n^{2-s}$.

By Lemmas 1 and 2, we have immediately the following theorem.

Theorem 1. *The measure in Lemma 2*

$$\mu_s=\sum_{n=1}^{+\infty} n^{m-s}\mu_n$$

is an element of $\mathcal{G}(\phi_N)$ *with infinite norm.*

Remark. H. Cartan [2] characterised the. class of measures $\mathcal{G}(\phi_N)$: A measure μ is an element of $\mathcal{G}(\phi_N)$ if and only if $\phi_N\mu(x) \not\equiv +\infty$. The above theorem shows that there are infinitely many positive measures μ_s with infinite norm that $\phi_N\mu_s(x)\not\equiv +\infty$.

2. Existence of a measure $\mu \in \mathcal{G}(\phi_\alpha)$ with infinite norm.

The α-kernel $\phi_\alpha(x, y)$ in R^m is defined by

$$\phi_\alpha(x, y)=|x-y|^{\alpha-m}$$

where α is any real number such as $0<\alpha<m$. O. Frostman [3] studied deeply the α-potential and proved many remarkable theorems. Above all, we start from his following theorem: Given a closed region F of which the boundary satisfies the Poincaré's condition, there exists a positive measure γ with unit mass and supported by F of which the potential $\phi_\alpha\gamma(x)$ is a positive constant V on F and is continuous in R^m. We shall say such a measure the equilibrium measure on F. This shows that $\mathcal{F}(\phi_\alpha)$ is not empty and we can consider the class of measures

$$\mathcal{U}(\phi_\alpha)=\{\textit{Equilibrium measure on all balls in } R^m\}.$$

Especially the equilibrium measure on the ball $B_{a,r}$ is denoted by $\gamma_{a,r}$.

Lemma 3. *For a given positive measure* μ, *the mutual energy* $\int\phi_\alpha\mu d\nu$ *is finite for any* $\nu \in \mathcal{F}(\phi_\alpha)$ *if* $\int\phi_\alpha\mu d\gamma$ *is finite for any* $\gamma \in \mathcal{U}(\phi_\alpha)$.

Proof. By the analogous way in the demonstration of Lemma 1, we can choose such a measure $\gamma \in \mathcal{U}(\phi_\alpha)$ that, for a suitable positive number t, $t\phi_\alpha\gamma(x)\geqq\phi_\alpha\nu(x)$ in R^m, because the α-kernel also satisfies the maximum and domination principles.

Lemma 4. *For any index* α *such as* $0<\alpha\leqq 2$, *any positive number* t *and any measure* $\gamma \in \mathcal{U}(\phi_\alpha)$, *there exists such a spherical distribution with uniform density* λ *that* $\phi_N\lambda(x)\geqq t\phi_\alpha\gamma(x)$ *in* R^m.

Proof. For any index α such as $0<\alpha\leqq 2$, the α-potential of a positive measure ν is subharmonic in the complementary set of the support $S\nu$. On the other hand, the Newtonian potential of a positive measure is superharmonic in R^m. So, in order to prove this lemma, it is sufficient to show that, for any positive number t and any measure $\gamma_{a,r} \in \mathcal{U}(\phi_\alpha)$, there exists a suitable measure $\lambda \in \mathcal{S}$ that $\phi_N\lambda(x) \geqq t\phi_\alpha\gamma_{a,r}(x)$ on the sphere $S_{a,r}$, the boundary of $B_{a,r}$. Owing to (*) and choosing $\lambda_{a,r} \in \mathcal{S}$ with a sufficiently large total mass M, we can make the value of $\phi_N\lambda_{a,r}(x)$ on $S_{a,r}$ larger than that of $t\phi_\alpha\gamma_{a,r}(x)$ on the same sphere.

By Lemmas 2, 3 and 4, we have the following theorem.

Theorem 2. *The measure in Lemma* 2

$$\mu_s=\sum_{n=1}^{+\infty} n^{s-m}\mu_n$$

is also an element of $\mathcal{G}(\phi_\alpha)$ $(0<\alpha\leqq 2)$ *with infinite norm.*

References

[1] G. Anger: Funktionalanalytische Betrachtungen bei Differential gleichungen unter Verwendung von Methoden der Potentialtheorie. I. Akademie-Verlag, Berlin (1967).

[2] H. Cartan: Théorie générale du balayage en potentiel newtonien. Ann. Univ. Grenoble, **22** (1946).

[3] O. Frostman: Potentiel d'équilibre et capacité des ensembles. Lund (1935).