194. *Dimension of Dispersed Spaces*

By Keiô NAGAMI

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1970)

Telgársky [5] showed that if X is a paracompact dispersed space, then $\operatorname{ind} X = \dim X = \operatorname{Ind} X = 0$. In this paper we consider the equalities between dimension functions defined on hereditarily paracompact spaces which are dispersed by some classes of spaces. All spaces in this paper are Hausdorff.

Let P be a property such that if a space X has P, then each closed subspace of X has P too. P need not be a topological one. Let $\mathcal{C}$ be the class of all spaces with P. A space X is said to be dispersed by $\mathcal{C}$, to be $\mathcal{C}$-dispersed or to be P-dispersed, if each non-empty closed set of X contains a point x one of whose relative neighborhoods is an element of $\mathcal{C}$. Let Y be a subset of X and Y' the set of all points y in Y one of whose relative neighborhoods is an element of $\mathcal{C}$. Set $Y^{(0)} = Y$, $Y^{(1)} = Y - Y'$ and $Y^{(\alpha)} = \cap\{(Y^{(\beta)})^{(1)} : \beta < \alpha\}$ for an ordinal $\alpha > 0$. Each $X^{(\alpha)}$ is closed. X is $\mathcal{C}$-dispersed if and only if $X^{(\gamma)} = \emptyset$ for some ordinal γ. If X is $\mathcal{C}$-dispersed, then an ordinal-valued function d on X is defined: $d(x) = \alpha$ if and only if $x \in X^{(\alpha)} - X^{(\alpha+1)}$. Let $d(X)$ denote the minimal ordinal α such that $X^{(\alpha)} = \emptyset$.

Theorem 1. *Let X be a hereditarily paracompact space. Then the following are true.*

i) *If X is metric-dispersed, then* $\dim X = \operatorname{Ind} X$.

ii) *If X is separable-metric-dispersed, then* $\operatorname{ind} X = \dim X = \operatorname{Ind} X$.

Proof (by transfinite induction on $d(X)$). Consider the case i). Put the induction assumption that the assertion is true for each hereditarily paracompact space Y with $d(Y) < d(X)$. When $d(X) = 1$, X is locally metric. Hence the whole X is metric by its paracompactness and the equality $\dim X = \operatorname{Ind} X$ is assured by well known Katětov-Morita's theorem. When $d(X) = \alpha + 1$ and $\alpha > 0$, then $(X - X^{(\alpha)})^{(\alpha)} = \emptyset$. Thus $d(X - X^{(\alpha)}) \leq \alpha$ and $\dim(X - X^{(\alpha)}) = \operatorname{Ind}(X - X^{(\alpha)})$ by the induction assumption. Since $\dim X = \max\{\dim X^{(\alpha)}, \dim(X - X^{(\alpha)})\}$ (cf. e.g. Nagami [3, Theorem 9–11]) and $\operatorname{Ind} X = \max\{\operatorname{Ind} X^{(\alpha)}, \operatorname{Ind}(X - X^{(\alpha)})\}$ (cf. Dowker [1, Theorem 3]), we have $\dim X = \operatorname{Ind} X$. When $d(X)$ is the limit ordinal, for each point x of X, $d(x) + 1 < d(X)$. Set $V(x) = X - X^{(d(x)+1)}$. Then $V(x)$ is an open neighborhood of x with $V(x)^{d(x)} = \emptyset$. Hence $\dim V(x) = \operatorname{Ind} V(x)$ by the induction assumption. Since $\dim X = \sup\{\dim V(x) : x \in X\}$ (cf. e.g. Dowker [2, Theorem 3.3]) and $\operatorname{Ind} X$

$=\sup\{\operatorname{Ind} V(x): x \in X\}$ (cf. Dowker [2, Theorem 3.4]), we have $\dim X = \operatorname{Ind} X$. The induction is completed.

The case ii) is verified analogously, starting from the equality $\operatorname{ind} X = \dim X = \operatorname{Ind} X$ for a separable metric X. The proof is finished.

Let C_1 be the class of all metric-dispersed spaces. Then we can define C_1-dispersed spaces. For such a space X we may have $\dim X = \operatorname{Ind} X$ if X is hereditarily paracompact. But we cannot get a wider category of spaces in this manner as the following shows.

Theorem 2. *If a space X is dispersed by the class of C-dispersed spaces, then X itself is C-dispersed.*

Proof. Let F be an arbitrary non-empty closed set of X. Then F contains a point x whose relative closed neighborhood H is C-dispersed. Since P is hereditary to closed subsets, we can assume without loss of generality that H is the closure of a relative open neighborhood U of x. Let y be a point of H and V an open set of H such that $y \in V$ and $\bar{V}$ has the property P. Then $U \cap V$ is a non-empty open set of H such that $\overline{U \cap V}$ has the property P. Thus X is C-dispersed and the theorem is proved.

Here is another way to get a space X holding $\dim X = \operatorname{Ind} X$: If X is a hereditarily paracompact space which is the countable sum of closed metric sets, then $\dim X = \operatorname{Ind} X$. The class of this type of spaces, say C_2, covers somewhat complementary domain to the class of metric-dispersed spaces, say C_3. But both C_2 and C_3 have the same feature (not so good feature) as this: They are not countably productive but finitely productive (cf. Nagami [4, Theorem 1]). An infinite full polyhedron with the weak topology is in C_2 and not in C_3. It is hereditarily paracompact. An example X which is in C_3 and not in C_2 is as follows. Let Y be the topologically disjoint sum of uncountably many metric spaces $X_\lambda, \lambda \in \Lambda$. Let X be the sum of Y and a single point p. Each open set of Y is open in X. A basic neighborhood of p is the set of the type: X minus the finite sum of X_λ's. Then X is hereditarily paracompact space which is not in C_2 but in C_3.

References

[1] C. H. Dowker: Inductive dimension of completely normal spaces. Quart. J. Math. Oxford, **4** (2), 267–281 (1953).

[2] ——: Local dimension of normal spaces. Quart. J. Math. Oxford, **6** (2), 101–120 (1955).

[3] K. Nagami: Dimension Theory. Academic Press, New York (1970).

[4] ——: Dimension for σ-metric spaces (to appear).

[5] R. Telgársky: Total paracompactness and paracompact dispersed spaces. Bull. Acad. Polon. Sci. Ser. Math., **16**, 567–572 (1968).