218. *On a Characterization of a Potential Theoretic Measure*

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Introduction. In the first place, G. Anger [1] pointed out that continuous potentials play an important role in the theory of potential. We are concerned with a kernel $\phi(x,y)$ with continuous potentials in a locally compact space. In the case, we can define a certain family of potential theoretic positive measures $G^+(\phi)$ of which the adjoint potentials are integrable by all measures generating continuous potentials. The aim of this paper is to characterize the family of measures $G^+(\phi)$, which answers at the same time for a question posed by G. Anger [2] in the case that ϕ are Newtonian kernel Φ_N and a kernel Φ_W associated with the fundamental solution of the heat equation. For the Newtonian kernel Φ_N, H. Cartan [4] gave the following well known result; In order that a positive measure μ is an element of $G^+(\Phi_N)$, it is necessary and sufficient that the potential of μ is not identically infinity. But the above result does not hold for the kernel Φ_W and then we must find another characterization.

1. Notations and definitions. Let Ω be a locally compact Hausdorff space and $\phi(x,y)$ a measurable function in $\Omega\times\Omega$. The kernel $\check{\phi}(x,y)$ defined by $\check{\phi}(x,y)=\phi(y,x)$ is called the adjoint kernel. Setting $\phi^+(x,y)=\sup(\phi(x,y),0)$ and $\phi^-(x,y)=-\inf(\phi(x,y),0)$, we can denote $\phi(x,y)=\phi^+(x,y)-\phi^-(x,y)$. The ϕ-potential of a positive Radon measure μ in Ω is defined by

$$\phi\mu(x)=\int^*\phi(x,y)d\mu(y),$$

provided that $\phi\mu^+(x)$ and $\phi\mu^-(x)$ are not infinity at the same time. The adjoint potential $\check{\phi}\mu(x)$ is defined by the analogous way. $\phi(x,y)$ is called S-kernel if there exists at least such a positive measure λ that the support $S\lambda$ is compact and the potentials $\phi\lambda^+(x)$ and $\phi\lambda^-(x)$ are continuous in Ω. In the case that $\phi(x,y)$ is S-kernel, we can consider the following classes of measures,

$$F^+(\phi)=\{\lambda\,;\,\lambda\geqq 0, S\lambda \textit{ compact}, \phi\lambda^+(x) \textit{ and } \phi\lambda^-(x) \textit{ continuous in } \Omega\}$$

$$G^+(\phi)=\left\{\mu\,;\,\mu\geqq 0, \int^*\check{\phi}\mu^+d\lambda \textit{ and } \int^*\check{\phi}^-\mu d\lambda<+\infty \textit{ for any } \lambda\in F^+(\phi)\right\}.$$

$\phi(x,y)$ is called T-kernel if $\phi(x,y)$ is non-negative S-kernel and for

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any compact set K, there exist such a point x_K in Ω, a relatively compact open set U_K containing K and a positive constant M_K depending on x_K and U_K that $\check{\phi}(x,y)\leqq M_K\check{\phi}(x_K,y)$ for any x of K and any y of CU_K, where CU_K denotes the complementary set of U_K. For a T-kernel ϕ and a compact set K, we shall denote by E_K the set of the all points x_K with the above properties.

2. Characterization of the family $G^+(\phi)$.

Theorem 1. *Suppose that $\phi(x,y)$ is a T-kernel in Ω. If a non-negative measure μ is such a measure that, for any compact set K in Ω, there exists a point x_K in E_K that $\check{\phi}\mu(x_K)<+\infty$, μ is an element of $G^+(\phi)$. If for any compact set K there exists a positive measure λ of $F^+(\phi)$ of which the support $S\lambda$ is contained by E_K, then the converse holds.*

Proof. As the support $S\lambda$ is compact for any λ of $F^+(\phi)$ and the kernel $\phi(x,y)$ is T-kernel, for the compact set $S\lambda$, there exist such a point x_λ in Ω, a relatively compact open set U_λ containing $S\lambda$ and a positive constant M_λ depending on x_λ and U_λ that

$$\check{\phi}(x,y)\leqq M_\lambda\check{\phi}(x_\lambda,y) \text{ for any } x \text{ of } S\lambda \text{ and any } y \text{ of } CU_\lambda.$$

Let μ_{U_λ} and μ_{CU_λ} be the restrictions of μ on the set U_λ and CU_λ respectively. Then we have

$$\int^*\check{\phi}\mu d\lambda=\int^*\phi\lambda d\mu_{U_\lambda}+\int^*\phi\lambda d\mu_{CU_\lambda}.$$

U_λ being relatively compact and $\phi\lambda$ continuous in Ω, the first integral of the right hand side is finite. If x_λ is such a point that $\phi_\mu(x_\lambda)<+\infty$, the second integral is also finite, because we have

$$\begin{aligned}\int^*\phi\lambda d\mu_{CU_\lambda}&=\int^*\int^*\check{\phi}(x,y)d\mu_{CU_\lambda}(y)d\lambda(x)\\&\leqq\int^*\int^*M_\lambda\check{\phi}(x_\lambda,y)d\mu_{CU_\lambda}(y)d\lambda(x)\\&\leqq M_\lambda\int^*\check{\phi}_\mu(x_\lambda)d\lambda(x)\\&<+\infty.\end{aligned}$$

Consequently, if the assumptions are fulfiled, the integral $\int^*\check{\phi}\mu d\lambda$ is finite for any λ of $F^+(\phi)$, that is, μ is an element of $G^+(\phi)$. Suppose that μ is an element of $G^+(\phi)$. If there exists such a compact set K that $\check{\phi}_\mu(x_K)=+\infty$ for all x_K of E_K, we have, for a positive measure λ of $F^+(\phi)$ of which the support $S\lambda$ is contained by E_K,

$$\int^*\check{\phi}_\mu d\lambda=+\infty,$$

which is contradictory.

Let $\varphi_N(x)$ and $\varphi_W(x)$ be the fundamental solutions of $\Delta_n u=0$ and $\Delta_{n-1}u-\partial u/\partial x_n=0$ respectively, where Δ_n denotes the Laplacian in the

n-dimensional Euclidian space R^n. We define the following two kernels in R^n,

$$\Phi_N(x,y) = \varphi_N(x-y) = \begin{cases} \dfrac{1}{(n-2)\omega_n}\dfrac{1}{|x-y|^{n-2}} & \text{for } x \neq y \\ +\infty & \text{for } x=y \end{cases}$$

where ω_n is the surface area of the unit ball in $R^n (n \geqq 3)$ and

$$\Phi_W(x,y) = \varphi_W(x-y) = \begin{cases} \left(\dfrac{1}{2\sqrt{\pi(x_n-y_n)}}\right)^{n-1}\exp\left[-\dfrac{\sum_{i=1}^{n-1}(x_i-y_i)^2}{4(x_n-y_n)}\right] & \text{for } x_n > y_n \\ 0 & \text{for } x_n \leqq y_n \end{cases}$$

where x_i is the i-th coordinate of $x=(x_1, x_2, \cdots, x_n)$.

The adjoint kernel $\check{\Phi}_W$ is defined by $\check{\Phi}_W(x,y)=\Phi_W(y,x)$. Given a positive Radon measure μ, we define the potentials $\Phi_N\mu(x)$, $\Phi_W\mu(x)$ and $\check{\Phi}_W\mu(x)$ by the integrals

$$\Phi_N\mu(x)=\int\Phi_N(x,y)d\mu(y),$$

$$\Phi_W\mu(x)=\int\Phi_W(x,y)d\mu(y)$$

and

$$\check{\Phi}_W\mu(x)=\int\check{\Phi}_W(x,y)d\mu(y)$$

respectively.

Let $\mathcal{H}_U$ be the set of all solutions u in an open set $U \subset R^n$ of $\Delta_n u=0$ or $\Delta_{n-1}u+\partial u/\partial x_n=0$. The sheaf $\mathcal{H}$; $U \to \mathcal{H}_U$ satisfies Bauer's axioms and then R^n is a harmonic space associated with the above sheaf $\mathcal{H}$. In this case, all constants are harmonic. In the case of $\Delta_n u=0$, all open balls

$$B_{a,r}=\left\{x=(x_1, x_2, \cdots, x_n) \in R^n;\ \left(\sum_{i=1}^{n}(x_i-a_i)^2\right)^{1/2} < r \text{ for any } a=(a_1, \cdots, a_n) \text{ and } r>0\right\}$$

from a regular base and in the case of $\Delta_{n-1}u+\partial u/\partial x_n=0$, all cones

$$\check{\Delta}_r^a=\left\{x=(x_1, x_2, \cdots x_n) \in R^n;\ \left(\sum_{i=1}^{n-1}(x_i-a_i)^2\right)^{1/2} < x_n-a_n,\ a_n < x_n < a_n+r,\ \text{for any } a=(a_1, a_2, \cdots, a_n) \text{ and } r>0\right\}$$

from a regular base. We denote by μ_x^V the harmonic measure for any regular open set V and any $x \in V$. Especially, it must remark that the support $S\mu_x^V$ of harmonic measure μ_x^V in the case of $V=\check{\Delta}_r^a$ is the compact set

$$S\mu_x^V=\partial V \cap \{y \in R^n;\ y_n \geqq x_n\},$$

where ∂V is the boundary of V. In the axiomatic theory, a numerical function $S(x)$ in an open set U is called superharmonic if $S(x)$ satisfies the following conditions; 1) $-\infty < S(x) \leqq +\infty$ in U, 2) $S(x)$ is lower semi-continuous in U, 3) $S(x)$ takes finite value in a dense subset of U and 4) $S(x) \geqq \int S(y) d\mu_x^V(y)$ for any V in U and any $x \in V$.

The following lemma is proved by H. Bauer [3] and J. L. Doob [5].

Lemma 1. *If μ is a positive Radon measure with compact support, then the potentials $\Phi_N \mu(x)$ and $\check{\Phi}_W \mu(x)$ are superharmonic in R^n.*
The minimum principle is the most important property of the superharmonic function [3].

Lemma 2. *Suppose that all constants are harmonic. Let $\tilde{U}$ be a compactification of a harmonic space U and $S(x)$ a superharmonic function in U. If it holds that $\liminf_{x \to y} S(x) \geqq 0$ for all $y \in \tilde{U} - U$, then we have $S(x) \geqq 0$ in U.*

It is well known that Φ_N and Φ_W are non-negative S-kernel and, now, on the base of Lemmas 1 and 2, we can prove that both kernels are T-kernel.

Lemma 3. *The kernel Φ_N is a T-kernel and, for any compact set K, the set E_K is identified with the whole space R^n.*

Proof. For any compact set K, we take any point x_0 of R^n and any relatively compact open set U_K which contains K and of which the boundary ∂U_K does not contain x_0. $\Phi_N(x, y)$ being positive and continuous if $x \neq y$ and ∂U_K compact, there exist the following minimum and maximum values,

$$\min \Phi_N(x_0, y) = \alpha > 0 \qquad \text{for any } y \in \partial U_K$$
$$\max \Phi_N(x, y) = \beta > 0 \qquad \text{for any } x \in K \text{ and } y \in \partial U_K$$

Choosing such a suitable positive constant M_K that $M_K \alpha \geqq \beta$, we have the following inequality,

$$M_K \Phi_N(x_0, y) \geqq \Phi_N(x, y) \qquad \text{for any } x \in K \text{ and any } y \in \partial U_K.$$

For any $x \in K$, the function $M_K \Phi_N(x_0, y) - \Phi_N(x, y)$ is superharmonic in $C\bar{U}_K$ and it holds that

$$\liminf_{y \to z} \{M_K \Phi_N(x_0, y) - \Phi_N(x, y)\} \geqq 0 \text{ for any } z \in \widetilde{C\bar{U}_K} - C\bar{U}_K,$$

where $\widetilde{C\bar{U}_K}$ is a compactification of $C\bar{U}_K$. Owing to Lemma 2, we have

$$M_K \Phi_N(x_0, y) \geqq \Phi_N(x, y) \text{ for any } x \in K \text{ and } y \in CU_K.$$

This shows that Φ_N is a T-kernel. In this proof, x_0 is an arbitrary point of R^n. So, for any compact set K, E_K is identified with the whole space R^n.

Lemma 4. *The kernel Φ_W is a T-kernel.*

Put $a_K = \min_{x \in K}$ {the n-th coordinate of x} for any compact set K. Then, for a compact set K, E_K contains the following set,

$$F_K = \{x = (x_1, x_2, \cdots, x_n) \in R^n;\ x_n \leqq b_K \text{ for any } b_K < a_K\}.$$

Proof. For any compact set K, we take any point x_K of F_K and any relatively compact open set $U_K \supset K$ of which the closure $\bar{U}_K$ is disjointed with F_K. $\check{\Phi}_W(x_K, y)$ is positive and continuous for any $y \in \partial U_K$ and $\check{\Phi}_W(x, y)$ is continuous and not identically zero for any $x \in K$ and any $y \in \partial U_K$, so there exist the following minimum and maximum values,

$$\min \check{\Phi}_W(x_K, y) = \gamma > 0 \qquad \text{for any } y \in \partial U_K$$

$$\max \check{\Phi}_W(x, y) = \delta > 0 \qquad \text{for any } x \in K \text{ and any } y \in \partial U_K$$

If we choose such a positive constant M_K as $M_K \gamma \geqq \delta$, then we have the following inequality,

$$M_K \check{\Phi}_W(x_K, y) \geqq \check{\Phi}_W(x, y) \qquad \text{for any } x \in K \text{ and any } y \in \partial U_K.$$

By Lemma 1, $\Phi_W(x_K, y)$ is superharmonic in R^n and, for any $x \in K$, $\check{\Phi}_W(x, y)$ is harmonic in $C\bar{U}_K$, because it holds that $\Delta_{n-1}\check{\Phi}_W + \partial \check{\Phi}_W / \partial y_n = 0$ in $C\bar{U}_K$. Owing to Lemma 2, we have

$$M_K \check{\Phi}_W(x_K, y) \geqq \Phi_W(x, y) \qquad \text{for any } x \in K \text{ and any } y \in CU_K$$

Preceding proof is valid for any point x_K of F_K, so E_K contains the set F_K but it must remark that E_K does not contain the set

$$\{x = (x_1, x_2, \cdots, x_n) \in R^n ; x_n \geqq d_K \textit{ for any } d_K > c_K\},$$

where c_K denotes $\max_{x \in K} \{\textit{the n-th coordinate of } x\}$.

Now we have the following characterization of the families $G^+(\Phi_N)$ and $G^+(\Phi_W)$.

Theorem 2. *In order that a non-negative measure μ is an element of $G^+(\Phi_N)$ (resp. $G^+(\Phi_W)$), it is necessary and sufficient that for any compact set K, there exist at least such a point x_K of E_K that $\Phi_N \mu(x_K)$ (resp. $\check{\Phi}_W \mu(x_K)$) is finite.*

Proof. By Lemmas 3 and 4, Φ_N and Φ_W are T-kernel and E_K contains an open set. For the kernel Φ_N (resp. Φ_W), any open set U contains the support $S\lambda$ of a positive measure λ of $F^+(\Phi_N)$ (resp. $F^+(\Phi_W)$). For instance, we can take as the above λ a harmonic measure μ_x^V of which the support $S\mu_x^V$ is contained by U. Applying Theorem 1, we have immediately Theorem 2.

Remark. For the Newtonian kernel Φ_N, H. Cartan [4] gave the following well known result; In order that a positive measure μ is an element of $G^+(\Phi_N)$, it is necessary and sufficient that the potential $\Phi_N \mu(x)$ is not identically infinity.

The following example shows that the above result does not hold for the kernel Φ_W. Let K be a compact set containing an open set in R^n and D an unbounded set $\{x = (x_1, x_2, \cdots, x_n) \in R^n ; b < x_n < c \textit{ for any } b \textit{ and } c \textit{ such as } a_K < b < c\}$ where $a_K = \max_{x \in K} \{\textit{the n-th coordinate of } x\}$.

Denote by $f(y)$ the positive function $1 / \min_{x \in K} \check{\Phi}_W(x, y)$ for any y of D and by μ a measure with the density function $f(y)$. Then the potential

$\check{\Phi}_W\mu(x)$ is plus infinity in K but zero at any point x of the set $\{x=(x_1, x_2, \cdots x_n) \in R^n; c \leqq x_n\}$. Consequently, the potential $\check{\Phi}_W\mu(x)$ is not identically plus infinity but is not an element of $G^+(\Phi_W)$, because it holds that

$$\int \check{\Phi}_W \mu d\lambda = +\infty$$

for a positive measure λ of $F^+(\Phi_W)$ of which the support S_λ is contained by K.

References

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