216. *Neutron Transport Process on Bounded Homogeneous Domain*

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1. The neutron transport process has been studied by Harris ([1]) and Mullikin ([5]) as an application of the theory of discrete-time branching processes. The main problems are the asymptotic behavior of the number of neutrons, the extinction probability and the rate of convergence of the extinction probability at time t to the extinction probability. In this paper we consider similar problems for a mono-energetic and isotropic neutron transport process on a bounded homogeneous domain. We will formulate the model as a continuous-time branching process and apply the general theory of such processes ([2]). Main results are the theorems $1 \sim 5$ below. It will be seen that the expected number of new-born neutrons plays an essential role in the above problems. This is a typical property of branching processes, which is well known for Galton-Watson processes.

2. Let D be a bounded closed convex domain in the three-dimensional Euclidian space $\boldsymbol{R}^3$ with a smooth boundary and Ω be the unit sphere in $\boldsymbol{R}^3$. We denote by G the product space $D \times \Omega$ and ∂G the set (x, ω) where x belongs to the boundary of D and ω is a direction exiting the domain; i.e., $(\omega, n_x) \geqq 0$ where n_x is the direction of the outer-normal at x. We formulate our model of neutron transport process as a continuous-time branching process as follows; a particle at $x \in D$ starting with unit speed in the direction ω*) will, at a random time T which is exponentially distributed with mean σ^{-1}, be absorbed, scattered, or multiplied by fission. If it leaves the domain D before T, then it is absorbed. The direction of new particles is supposed to be isotropically distributed. Each of new particles, independently each other, performs a similar motion as the original one. We can construct such a branching process on a suitable probability space ([2]) and every probabilistic argument below is based on this process.

Let $F[\xi] = \sum_{n=0}^{\infty} p_n \xi^n$ where p_n is the probability that n neutrons are produced when fission occurs. (In particular p_0 is the probability of absorption and p_1 the probability of scattering.) We will assume $F'[1] < \infty$ and $p_0 + p_1 < 1$. The first assumption guarantees that the

*) This statement will be simplified below as "starting at (x, ω)."

explosion does not occur in finite time. By the general theory, the extinction probability $q(t, x, \omega)$ at time t starting at (x, ω) is the unique solution of

$$(1)\quad \begin{cases} \dfrac{\partial u(t, x, \omega)}{\partial t} = \omega \cdot \nabla u(t, x, \omega) - \sigma u(t, x, \omega) + \sigma F[\tilde{u}(t, x)] (\equiv Au(t, x, \omega)) \\ u(t, x, \omega) = 1,\ (x, \omega) \in \partial G \\ u(0+, x, \omega) = 0. \end{cases}$$

Here, "$\sim$" means the direction average; $\tilde{u}(t, x) = \dfrac{1}{4\pi}\displaystyle\int_{\Omega} u(t, x, \omega) d\omega$.

The extinction probability $q(x, \omega)$ starting at (x, ω), which is given as $\lim_{t\to\infty} q(t, x, \omega)$, is the smallest solution of

$$(2)\qquad Au = 0 \text{ in } G,\ u(x, \omega) = 1 \text{ if } (x, \omega) \in \partial G,\ 0 \leqq u(x, \omega) \leqq 1.$$

This equation has a trivial solution $u(x, \omega) \equiv 1$. Let $E(x, y\,; \lambda)$ be the function $\sigma e^{-(\sigma+\lambda)|x-y|}/4\pi|x-y|^2$ on $D \times D$ for each complex λ, and c^{*-1} be the largest positive eigenvalue of the operator induced by the kernel $E(x, y\,; 0)$ on the Banach space $\boldsymbol{C}(D)$ of all continuous functions on D with sup-norm.

Lemma 1 (Pazy-Rabinowitz [6]). *If $F'[1] \leqq c^*$, then* (2) *has no non-trivial solution and hence $q(x, \omega) \equiv 1$. If $F'[1] > c^*$, it has the unique non-trivial solution which is, therefore, equal to $q(x, \omega)$. Furthermore, $q(x, \omega) < 1$, if $(x, \omega) \in G - \partial G$, and $\inf_{(x,\omega)\in G} q(x, \omega) > 0$.*

3. In this section we shall consider the following equation.

$$(3)\quad \begin{cases} \dfrac{\partial u(t, x, \omega)}{\partial t} = \omega \cdot \nabla u(t, x, \omega) - \sigma u(t, x, \omega) + k(x)\tilde{u}(t, x) (\equiv Bu(t, x, \omega)) \\ u(t, x, \omega) = 0 \text{ if } (x, \omega) \in \partial G \\ u(0+, x, \omega) = f(x, \omega) \end{cases}$$

where $k(x)$ is continuous on D, bounded, and bounded away from 0 by a positive constant. Let $\boldsymbol{C}_0(G)$ be the Banach space of all continuous functions on G which vanish on ∂G with sup-norm $\|\cdot\|$, and $\boldsymbol{H}(G)(\boldsymbol{H}(D))$ be the Hilbert space of all spuare integrable functions with L^2-norm $\|\cdot\|_2$ on G (resp. D). First we shall consider the following eigenvalue problem in the space $\boldsymbol{C}_0(G)$ or $\boldsymbol{H}(G)$.

$$(4)\qquad B\psi = \lambda\psi, \qquad \psi(x, \omega) = 0 \text{ if } (x, \omega) \in \partial G.$$

If ψ is the solution of (4), then $\varphi(x) = \tilde{\psi}(x)$ satisfies

$$(5)\qquad \varphi(x) = \int_D E(x, y\,; \lambda)\varphi(y)k(y)dy (\equiv E(\lambda)\varphi(x))$$

Conversely, if $\varphi \in \boldsymbol{C}(D)$ is the solution of (5), then there exists a unique solution ψ of (4) such that $\varphi = \tilde{\psi}$. S. Ukai has proved that there exist infinitely many real eigenvalues of B, the largest one is simple, and the corresponding eigenfunction is everywhere positive ([7]).

Lemma 2. *Let λ_0 be the real eigenvalue of B with maximal real part. Then there exist no eigenvalues of the form $\lambda_0 + ic$, $c \neq 0$.*

Proof. Suppose that $\lambda=\lambda_0+ic$ is a eigenvalue of B with corresponding eigenfunction $\psi_c(x,\omega)$. Since $\varphi_c=\bar{\psi}_c$ satisfies $\varphi_c=E(\lambda)\varphi_c$, we have $|\varphi_c|(x)\leqq E(\lambda_0)|\varphi_c|(x)$.

Assume $|\varphi_c|(x)\not\equiv E(\lambda_0)|\varphi_c|(x)$, then

$$(|\varphi_c|,\varphi)<(E(\lambda_0)|\varphi_c|,\varphi)=(|\varphi_c|,E(\lambda_0)^*\varphi)=(|\varphi_c|,\varphi)$$

Therefore $|\varphi_c|\equiv E(\lambda_0)|\varphi_c|$.

By virtue of the simplicity of φ_c,

$\varphi_c(x)=\varphi(x)e^{if(x)}$ and f is a continuous function on D.

From the definition

$$\varphi(x)=\int_D E(x,y;\lambda_0)\varphi(y)\exp\{i\{f(y)-f(x)-c|x-y|\}\}k(y)\,dy,$$

and observing that φ is real, we have

$$\varphi(x)=\int_D E(x,y;\lambda_0)\varphi(y)\cos\{f(y)-f(x)-c|x-y|\}k(y)dy.$$

By the definition of φ, we must have $\cos\{f(y)-f(x)-c|x-y|\}=1$. Since f is continuous, $f(y)-f(x)-c|x-y|=0$, but this is a contradiction if $c\neq 0$, and lemma is proved.

The largest eigenvalue $\mu(\beta)$ of $E(\beta)$ as a function of real β is continuous, and strictly decreasing for $\beta>-\sigma$, as shown by S. Ukai ([7]). The strict decreasing property of $\mu(\beta)$ for real β can be proved easily by using the next lemma.

Lemma 3 (Karlin [4]). *Suppose that E is completely continuous and strictly positive operator over a Banach space, then the largest eigenvalue r of E is given by*

$$r=\sup\{\lambda\,|\,{}^{\exists}x\neq 0,\ x\geqq 0,\ Ex\geqq\lambda x\}=\inf\{\lambda\,|\,{}^{\exists}x\neq 0,\ x\geqq 0,\ Ex\leqq\lambda x\}.$$

Lemma 4. *$\mu(B)$ is strictly decreasing.*

Proof. Let φ_β be the eigenfunction corresponding to $\mu(\beta)$. Since φ_β is everywhere positive on D, φ_β is bounded away from 0 by a positive constant, i.e. φ_β is a strictly positive element. We may assume that $\|\varphi_\beta\|=1$. Then for every positive ε, there exists a positive constant $\eta>0$ such that $\{E(\beta-\varepsilon)-E(\beta)\}\varphi_\beta\geqq\eta$ $\cdot$ Hence $E(\beta-\varepsilon)\varphi_\beta\geqq E(\beta)\varphi_\beta+\eta\geqq(\mu(\beta)+\eta)\varphi_\beta$, therefore $\mu(\beta-\varepsilon)\geqq\mu(\beta)+\eta>\mu(\beta)$ by Lemma 3.

Keeping the Jörgens' results ([3]) in mind, we can obtain the following

Lemma 5. *There exists a one-parameter semigroup M_t on $\boldsymbol{C}_0(G)$ or on $\boldsymbol{H}(G)$, such that $u(t,x,\omega)=M_tf(x,\omega)$ satisfies* (3). *Moreover there exist positive constants T_0 and ρ such that for every $t\geqq T_0$, and for every $f\in\boldsymbol{C}_0(G)$*

$$\|M_tf(x,\omega)-e^{\lambda_0t}(f,\psi^*)\psi(x,\omega)\|\leqq e^{\lambda_0t}\,0(e^{-\rho t}\|f\|)$$

where λ_0 is the eigenvalue of B with maximal real part, and $\psi(\psi^*)$ is the corresponding eigenfunction of B (resp. B^*). When f is in $\boldsymbol{H}(G)$, the same estimate holds if we replace only $\|f\|$ in the right-hand side by $\|f\|_2$.

If $k(x)\equiv\sigma F'[1]$ the solution $u(t,x,\omega)$ of equation (3) represents the expected number of neutrons at time t starting at (x,ω). Let α be the eigenvalue of B with maximal real part in this case. Then from strict decreasing property of $\mu(\beta)$, we have

Lemma 6. *$\alpha<0$, $\alpha=0$, or $\alpha>0$ according as $F'[1]<c^*$, $F'[1]=c^*$, or $F'[1]>c^*$, respectively.*

4. We shall study the asymptotic behavior of $r(t,x,\omega)=q(x,\omega)-q(t,x,\omega)$. Let α be as above, and $\psi(x,\omega)(\psi^*(x,\omega))$ be the corresponding eigenfunction of B (resp. B^*).

Theorem 1. *Suppose $F'[1]<c^*$, and $F''[1]<\infty$, then there exist positive constants C_1 and δ such that*

$$r(t,x,\omega)=C_1e^{\alpha t}\psi(x,\omega)+e^{\alpha t}0(e^{-\delta t}),\ t\to\infty.$$

Theorem 2. *Suppose $F'[1]=c^*$, and $F'''[1]<\infty$, then there exists a positive constant C_2 such that*

$$r(t,x,\omega)=C_2\psi(x,\omega)/t+0(1/t),\ t\to\infty$$

Theorem 3. *Suppose $F'[1]>c^*$, and $F''[1]<\infty$, then $q(x,\omega)\not\equiv1$ and there exist positive constants C_3 and ε such that*

$$r(t,x,\omega)=C_3e^{\gamma t}\bar{\psi}(x,\omega)+e^{\gamma t}0(e^{-\varepsilon t}),\ t\to\infty$$

where γ is the eigenvalue of B in the case $k(x)\equiv\sigma F'[\tilde{q}(x)]$, and $\bar{\psi}(x,\omega)$ is the corresponding eigenfunction. In this case, $\mu(0)<1$, and $\gamma<0$ from the strict increasing property of $\mu(\beta)$.

Let Z_t^E be the number of particles in $E\subset G$ at time t starting at (x,ω).

Theorem 4. *Suppose $F'[1]>c^*$, and $F''[1]<\infty$. Then there exists a non-negative random variable W such that*

$$\{W>0\}=\{Z_t^G\to\infty \text{ as } t\to\infty\}\quad \text{a.s.},$$

and for every $E\subset G$ such that $(I_E,1)$) >0,*

$$E[|Z_t^E\{e^{\alpha t}(I_E,\psi^*)\}^{-1}-W|^2]=0(e^{-\varepsilon t})$$

where ε is independent of E.

Theorem 5. *Suppose $F'[1]=c^*$, and $F''[1]<\infty$. Then for every $E_1,E_2,\cdots,E_n\subset G$ such that $(I_{E_i},1)>0$ $(i=1,2,\cdots,n)$, the joint distribution of $\{t2^{-1}\sigma F''[1](\psi^2,\psi^*)\}^{-1}(Z_t^{E_1},Z_t^{E_2},\cdots,Z_t^{E_n})$ under the condition $Z_t^G\neq0$, converges to that of $((I_{E_1},\psi^*),(I_{E_2},\psi^*),\cdots,(I_{E_n},\psi^*))\cdot W$, when $t\to\infty$, where W is exponentially distributed with mean 1.*

References

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*) I_E is the indicator function of the set E.

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