211. *On Semifield Valued Functions*

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In our Note [2], we generalized the well known Hahn-Banach theorem on the extension of linear functionals to semifield valued functions. In this Note, we concern with semifield valued linear functions, and generalize some results of R. P. Agnew [1]. As well known, the theory of semifields has been developed by M. Ya. Antonovski, B. G. Boltjanski and T. A. Sarymsakov since 1960. We shall use some results of semifields without references.

Let E be a linear space, and let F be a semifield. We consider functions defined on E with range in F. A function $r(x)$ is called r-function on E, if there is a linear function $f(x)$ satisfying $f(x) \ll r(x)$ on E.

Let $r(x)$ be an r-function on E. Then there is a linear function $f(x)$ such that $f(x) \ll r(x)$ for all $x \in E$. For $t>0$, we have

$$f(x)=\frac{f(tx)}{t} \ll \frac{r(tx)}{t}$$

Therefore, if $t_1, t_2, \cdots, t_n > 0$, and $\sum_{i=1}^{n} x_i = 0$, we have

$$0=f(x)=f\left(\sum_{i=1}^{n} x_i\right)=\sum_{i=1}^{n} f(x_i) \ll \sum_{i=1}^{n} \frac{r(t_i x_i)}{t_i},$$

hence

$$0 \ll \inf_{\substack{t_i>0 \\ \sum_{i=1}^{n} x_i=0}} \sum_{i=1}^{n} \frac{r(t_i x_i)}{t_i} \tag{1}$$

Conversely, suppose that (1) holds. We define

$$p(x)=\inf_{\substack{t_i>0 \\ x_i=x}} \sum_{i=1}^{n} \frac{r(t_i x_i)}{t_i}. \tag{2}$$

Then $p(x) \ll r(x)$. Let $\sum_{i=1}^{n} x_i = x$, then $\sum_{i=1}^{n} x_i + (-x) = 0$. By the hypothesis, we have

$$0 \ll \sum_{i=1}^{n} \frac{r(t_i x_i)}{t_i} + r(-x).$$

Hence $-r(-x) \ll p(x)$. If $x \in E$ and $t>0$, then

$$p(tx)=\inf_{\substack{t_i>0 \\ \sum_{i=1}^{n} x_i=x}} \sum_{i=1}^{n} \frac{r(t_i x_i)}{t_i}$$

$$=t \inf_{\substack{t_i>0 \\ \sum_{i=1}^n x_i/t=x}} \sum_{i=1}^n \frac{r(tt_i r_i/t)}{tt_i}$$

$$=t \inf_{\substack{s_i>0 \\ \sum_{i=1}^n y_i=x}} \sum_{i=1}^n \frac{r(s_i y_i)}{s_i}=tp(x).$$

If $t=0$, then $p(0)=0$. Hence for $t>0$, we have $p(tx)=tp(x)$. To prove the subadditivity of $p(x)$, i.e.

$$p(x+y)\ll p(x)+p(y), \tag{3}$$

it is sufficient to show that, for any neighborhood U of the zero in F, $p(x+y)\ll p(x)+U+p(y)+U$, by the general theory of semifields. For any neighborhood U of the zero in F, there are positive numbers $s_1, s_2, \cdots, s_m, t_1, t_2, \cdots, t_n$ and elements $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$ in E satisfying

$$\sum_{i=1}^m x_i=x, \sum_{i=1}^m \frac{r(s_i x_i)}{s_i}\ll p(x)+U,$$

$$\sum_{j=1}^n y_j=y, \sum_{j=1}^n \frac{r(t_j y_j)}{t_j}\ll p(y)+U.$$

From $\sum_{i=1}^m x_i+\sum_{j=1}^n y_j=x+y$, we have

$$p(x+y)\ll \sum_{i=1}^m \frac{r(s_i x_i)}{s_i}+\sum_{j=1}^n \frac{r(t_j y_j)}{t_j}\ll p(x)+U+p(y)+U,$$

hence (3) holds. By Hahn-Banach theorem mentioned in [2], there is a linear function $f(x)$ such that $f(x)\ll p(x)\ll r(x)$. This means that a function satisfying (1) is an r-function. Therefore we have the following

Theorem. *A function $r(x)$ on E is an r-function, if and only if*

$$\inf_{\substack{t_i>0 \\ \sum_{i=1}^n x_i=0}} \sum_{i=1}^n \frac{r(t_i x_i)}{t_i}\gg 0.$$

Theorem is a generalization of Agnew result, and its proof is essentially due to R. P. Agnew [1]. From Theorem, we have the following Corollary due to R. P. Agnew:

Corollary 1. *If $r(x)$ is an r-function, then $p(x)$ defined by* (2) *is a function such that*

$$p(x+y)\ll p(x)+p(y), \tag{3}$$

$$p(tx)=tp(x) \qquad \textit{for } t>0, \tag{4}$$

and $$-r(-x)\ll -p(x)\ll p(x)\ll r(x).$$

The function $p(x)$ is maximal, i.e., if $q(x)$ is a function of the type (3), (4) *and $q(x)\ll r(x)$ on E, then $q(x)\ll p(x)$.*

Corollary 2. *If $r(x)$ is an r-function and $f(x)$ is a linear function satisfying $f(x)\ll r(x)$ on E, then $f(x)\ll p(x)\ll r(x)$ on E.*

The proofs of two corollaries are quite similar with the case of real valued function.

References

[1] R. P. Agnew: On the existence of linear functionals defined over linear spaces. Bull. of Amer. Math. Soc., **43**, 868–872 (1937).
[2] K. Iséki and S. Kasahara: On Hahn-Banach type extension theorem. Proc. Japan Acad., **41**, 29–30 (1965).