210. *On the Generalized Korteweg-de Vries Equation*

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(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1970)

1. **Introduction.** The nonlinear dispersive equation of the type

$$u_t-(f(u))_x+\delta u_{xxx}=0, \qquad \delta>0 \tag{1.1}$$

is the generalization of the Korteweg-de Vries (KdV) equation

$$u_t-uu_x+\delta u_{xxx}=0, \qquad \delta\neq 0 \tag{1.2}$$

and is closely related to the study of anharmonic lattices, see [1].

In [2], A. Sjöberg proved, by the method of the semi-discrete approximation, existence and uniqueness of the global classical solutions of the KdV equation for an appropriate initial condition and periodic boundary condition. In [3], T. Mukasa and R. Iino extended Sjöberg's results to the simplest generalized KdV equation

$$u_t-u^2u_x+\delta u_{xxx}=0, \qquad \delta\neq 0. \tag{1.3}$$

By the method of parabolic regularization, R. Teman [4] obtained the existence and uniqueness theorems of the global weak solutions of the KdV equation for an appropriate initial condition and periodic boundary condition. In [5], Y. Kametaka proved existence and uniqueness of the global classical solutions of the Cauchy problem for the KdV equation and the simplest generalized KdV equation.
In [6], K. Masuda studied the Cauchy problem for the equation of the type

$$u_t-(u^p/p)_x+\delta u_{xxx}=0,\ p=1,2,3,4, \qquad \delta\neq 0. \tag{1.4}$$

In this note our aim is to extend those results to the generalized KdV equation (1.1) under the appropriate conditions imposed on $f(u)$, which include the cases $f(u)=u^p$, where p is an arbitrary odd number, and $f(u)=e^u$.

The plan of this paper is the following: In Section 2, we study the Cauchy problem for the nonlinear dissipative-dispersive equation

$$u_t-(f(u))_x+\delta u_{xxx}=\mu u_{xx}, \qquad \delta,\mu>0, \tag{1.5}$$

and establish the global existence theorems of the Cauchy problem for the equation (1.1) by letting μ tend to 0. Section 3 is devoted to present a semi-discrete approximation of the equation (1.5), which assures the global existence of the weak solutions of the initial-periodic boundary value problem for the equation (1.5). In Section 4, by the singular perturbation, we establish the existence theorem of the initial-periodic boundary value problem for the equation (1.1).

In this note we state the results only. Detailed proof will be published elsewhere.

2. Nonlinear dissipative-dispersive equation and the method of vanishing dissipations. Let $H^s(R)$[1] (s: integer) be the usual Sobolev space and norm $\|\cdot\|_s$. When there is no chance of confusion, we shall drop the symbol R. By $L^p(a, b; E)$ we denote the space of E-valued functions $u(t)$ on (a, b) for which

$$\left(\int_a^b \|u(t)\|_E^p dt\right)^{1/p} < \infty \qquad 1 \leqslant p < \infty$$
$$\sup_{a \leqslant t \leqslant b} \|u(t)\|_E < \infty \qquad p = \infty$$

where E is a Banach space with norm $\|\cdot\|_E$. We denote by $C^m[a, b; E]$,[2] the space of functions which are m times continuously differentiable over $[a, b]$ with values in E.

Now we consider the following Cauchy problem:

$$\text{(2.1)} \qquad \begin{cases} u_t - (f(u))_x + \delta u_{xxx} = \mu u_{xx}, & \delta, \mu > 0 \\ u(x, 0) = u_0(x). \end{cases}$$

Construct the sequence of approximate solutions $\{u^n(x, t)\}$ of the problem (2.1) as follows:

$$\text{(2.2)} \qquad \begin{cases} u_t^n - (f(\phi + u^{n-1}))_x + \delta u_{xxx}^n = \mu u_{xx}^n, & \delta, \mu > 0 \\ u^0(x, t) \equiv 0 \\ u^n(x, 0) = 0, & (n = 1, 2, \cdots\cdots) \end{cases}$$

where $\phi(x, t)$ is a solution of the problem

$$\text{(2.3)} \qquad \begin{cases} \phi_t + \delta \phi_{xxx} = \mu \phi_{xx}, & \delta, \mu > 0 \\ \phi(x, 0) = u_0(x). \end{cases}$$

Then we have the following local existence theorem.

Theorem 1. *Suppose that $f(u) \in C^{3(m+1)}(R)$. For every real-valued initial function $u_0(x) \in H^{3(m+1)}$ and for each $\delta, \mu > 0$, there exists a positive number T_m such that in the interval $0 \leqslant t \leqslant T_m$ the problem* (2.1) *has a unique solution $u(x, t)$ belonging to $L^\infty(0, T_m; H^{3(m+1)}) \cap C[0, T_m; H^{3m}] \cap \cdots \cap C^m[0, T_m; L^2]$.*

If we impose further assumptions on $f(u)$, then we have the following a periori estimates:

Theorem 2. *Suppose that f satisfies the condition of Theorem* 1 *and, furthermore, satisfies one of the following two conditions:*

Condition A $\quad df(u)/du \geqslant 0, \qquad \tilde{F}(u) = \int_0^u (f(v) - f(0))dv \geqslant 0,$

Condition B $\quad |f(u) - f(0)| \leqslant K_1(|u| + u^2 + |u|^3 + u^4),$

where K_1 is a positive constant.

Then the solution $u(x, t)$ of the problem (2.1) *satisfies a priori estimates of the form*

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1) In the sequel of this note, by the symbol R we always denote the one-dimensional real Euclidean space.

2) We always denote by m, a nonnegative integer.

$$\sup_{0\leqslant t\leqslant T}\|u\|_k\leqslant l(\|u_0\|_k), \qquad k=0,1,\cdots,3(m+1) \tag{2.4}$$

where T is an arbitrary positive number and l are positive valued monotone increasing functions with $l(0)=0$.

Combining Theorem 1 and Theorem 2, we easily conclude the global existence theorem:

Theorem 3. *Suppose that f satisfies the conditions of Theorem 2. If $u_0(x)\in H^{3(m+1)}$, then for every fixed $\delta,\mu>0$, the problem (2.1) has a unique solution $u(x,t)$ such that in any interval $0\leqslant t\leqslant T$,*

$$u(x,t)\in L^\infty(0,T;H^{3(m+1)})\cap C[0,T;H^{3m}]\cap\cdots\cap C^m[0,T;L^2]$$

As concerns the weak solutions of the problem (2.1), we have the following theorem:

Theorem 4. *Suppose that f satisfies the conditions of Theorem 2. If $u_0(x)\in H^i, i=1,2$, then the problem (2.1) has a unique weak solution $u(x,t)$ such that for any $T>0$*

$$u(x,t)\in L^\infty(0,T;H^i)\cap L^2(0,T;H^{i+1}).$$

Letting μ tend to 0, establish

Theorem 5. *Suppose that f satisfies the conditions of Theorem 2. If $u_0(x)\in H^i, i=1,2$, then there exists a function $u(x,t)$ which belongs to $L^\infty(0,T;H^i)$ for any $T>0$ and satisfies*

$$\begin{cases} u_t-(f(u))_x+\delta u_{xxx}=0, & \delta>0,\\ u(x,0)=u_0(x). \end{cases} \tag{2.5}$$

When $i=2$, the function $u(x,t)$ is uniquely determined by the initial function $u_0(x)$.

Theorem 6. *Suppose that f satisfies the conditions of Theorem 2. If $u_0(x)\in H^{3(m+1)}$, then the Cauchy problem (2.5) has a unique solution $u(x,t)$ such that for any $T>0$*

$$u(x,t)\in L^\infty(0,T;H^{3(m+1)})\cap C[0,T;H^{3m}]\cap\cdots\cap C^m[0,T;L^2].$$

Remark. In Theorem 3, 6, let $m=1$, then we establish the global existence theorems of the classical solutions of the problem (2.1) and (2.5).

3. Semi-discrete approximation. We consider the following initial-periodic boundary value ploblem:

$$\begin{cases} u_t-(f(u))_x+\delta u_{xxx}=\mu u_{xx}, & \delta,\mu>0\\ u(x,0)=u_0(x), & 0<x<1,\\ u(0,t)=u(1,t) & \text{for all } t\geqslant 0. \end{cases} \tag{3.1}$$

For an integer s, H_s is a Hilbert space whose elements are real-valued 1-periodic functions with finite norm

$$\|u\|_s^2=\sum(1+(2\pi k)^2)^s|\alpha_k|^2$$

where $\quad u(x)=\sum\alpha_k e^{2\pi ik}, \qquad \alpha_{-k}=\overline{\alpha}_k, \qquad i=\sqrt{-1}.$

If we develop our arguments in the function space H_s instead of the function space $H^s(R)$, the method mentioned in Section 2 is also applicable for the problem (3.1). Indeed, quite similar results as

Theorems 1–6 are obtained under the following conditions for $f(u)$: $f \in C^{3(m+1)}(R)$ satisfies one of the two conditions;

Condition A′ $\quad df(u)/du \geqslant 0$,

Condition B′ $\quad |f(u)| \leqslant K_2(1+u^4)$,

where K_2 is a positive constant. However, in this section we shall examine the problem (3.1) by the following semi-discrete approximation:

$$\text{(3.2)}\quad \begin{cases} du_N(x_r,t)/dt - D_0(f(u_N(x_r,t))) + \delta D_+D_-D_0u_N(x_r,t) \\ \qquad = \mu D_+D_-u_N(x_r,t), & r=1,2,\cdots,N \\ u_N(x_r,0)=u_0(x_r), & r=1,2,\cdots,N \\ u_N(x_r,t)=u_N(x_{r+N},t), & r=1,2,\cdots,N \text{ and all } t\geqslant 0 \end{cases}$$

where the mesh-width $h=1/N$, N natural number, $x_r=rh$ and the difference operators D_+, D_- and D_0 are defined by

$$hD_+u(x_r)=u(x_{r+1})-u(x_r), \qquad hD_-u(x_r)=u(x_r)-u(x_{r-1}),$$
$$2hD_0u(x_r)=u(x_{r+1})-u(x_{r-1}).$$

We construct a family of functions $\psi_N(x,t)$ from $u_N(x_r,t)$ in the following fashion:

$$\begin{aligned}\psi_N(x,t) = &-((x-x_r)^3/h)D_+D_-u_N(x_r,t)+2(x-x_r)^2D_+D_-u_N(x_r,t) \\ &+(x-x_r)D_-u_N(x_r,t)+u_N(x_r,t) \\ &\text{for } x_r \leqslant x < x_{r+1}.\end{aligned}$$

Then, making use of a priori estimates of (3.2) which are independent of h, we can show that $\psi_N(x,t)$ converges (as h tends to 0) to a function $u(x,t)$ which possesses the properties enunciated in the following theorem.

Theorem 7. *Suppose that $f \in C^1(R)$ satisfies the conditions*

$$df(u)/du > 0, \quad F(u)=\int_0^u f(v)dv \geqslant K_3|u|^2+K_4$$

where K_3 is a positive constant and K_4 is an arbitrary constant.

For $u_0(x) \in H_1$, there exists a unique function $u(x,t)$ which belongs to $L^\infty(0,T;H_1)\cap L^2(0,T;H_2)$ for any $T>0$ and satisfies the identity

$$-\int_0^T\int_0^1 u\eta_t dxdt + \int_0^T\int_0^1 f(u)\eta_x dxdt + \mu\int_0^T\int_0^1 u_x\eta_x dxdt$$
$$-\delta\int_0^T\int_0^1 u_{xx}\eta_x dxdt = \int_0^1 u_0(x)\eta(x,0)dx$$

for all test functions $\eta(x,t) \in C^\infty[0,T;H_\infty]$ that are equal to 0 at $t=T$. Here $H_\infty = \bigcap_s H_s$.

4. The method of the singular perturbation with a nonlinear term.

We consider the next equation with the additional terms consisting of the fourth order term and the nonlinear second order term:

$$\text{(4.1)}\qquad u_t-(f(u))_x+\delta u_{xxx}-\varepsilon((f(u))_{xx}-\delta u_{xxxx})=0.$$

Under the appropriate conditions imposed on $f(u)$ and initial value we can show the existence of a smooth solution $u_\varepsilon(x,t)$ of the initial-periodic boundary value problem for (4.1) by the semi-discrete approximation similar to the scheme presented in Section 3.

Letting $\varepsilon \to 0$ in u_ε, then we get a solution of the initial-periodic boundary value problem for the equation (1.1).

Theorem 8. i) *Suppose that* $f(u) \in C^{2m}(R)$ *satisfies the condition*

$$F(u)=\int_0^u f(v)dv \geqslant K_3|u|^2+K_4, \qquad K_3>0, \tag{4.2}$$

or the condition

$$df(u)/du \geqslant 0, \qquad F(u) \geqslant 0. \tag{4.3}$$

Then, for every $\varepsilon>0$ *and for initial value* $u_0(x) \in H_{2m}$, *there exists a unique solution* $u(x,t) \in L^\infty(0,T;H_{2m})$ *of* (4.1), *where* m, *integer* $\geqslant 4$.

ii) *Under the same conditions for* $f(u)$, *if* $u_0(x) \in H_k$, *then there exists a unique solution* $u(x,t) \in L^\infty(0,T;H_k)$ *of* (1.1), *where* k, *integer* $\geqslant 2$.

Remark. If we add the higher order dispersive term, for instance, the fifth order term, to the left member of the equation (1.1), we can also obtain the global existence and uniqueness theorems of the classical solutions of the Cauchy or initial-periodic boundary value problem for this equation under the appropriate conditions imposed on $f(u)$ and initial value.

Acknowledgement. The authors would like to express their sincere thanks to Mr. A. Yoshikawa for his informing Masuda's results.

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