16. *Integration of Alexander-Spanier Cochains*

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The purpose of this note is to define the notion of integration on singular chains for Alexander-Spanier cochains and state some of its properties such as Stokes' theorem. The notions of the volume element with respect to a metric and integral operator with its symbol for a (compact) CW complex are also given. The Details will appear in the Journal of the Faculty of Science, Shinshu University, Vol. 5, 1970.

0. Alexander-Spanier cochains. For a topological space X, we set

$$\Delta_s(X)=\{(x, x, \cdots, x) \mid x \in X\} \subset X \times \overset{s+1}{\cdots} \times X.$$

We denote by $\mathfrak{R}$ a topological vector space over $\boldsymbol{R}$ or $\boldsymbol{C}$.

Definition. Two $\mathfrak{R}$-valued functions f and g on $U(\Delta_s(X))$, a neighborhood of $\Delta_s(X)$, are called equivalent if

$$f \mid V(\Delta_s(X))=g \mid V(\Delta_s(X)),$$

for some neighborhood $V(\Delta_s(X))$ of $\Delta_s(X)$ and the equivalence class of f by this relation is called the germ of $f(at\ \Delta_s(X))$. The germ of f is denoted by $\bar{f}$ or simply, f.

Definition. A germ of f at $\Delta_s(X)$ is called an ($\mathfrak{R}$-valued) Alexander-Spanier s-cochain.

We call an Alexander-Spanier s-cochain $\bar{f}$ is continuous, regular or alternative if a representation f of $\bar{f}$ is continuous, $f(x_0, x_1, \cdots, x_s)=0$ if $x_i=x_j$ for some i, j or $f(x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(s)})=\operatorname{sgn}(\sigma) f(x_0, x_1, \cdots, x_s)$, $\sigma \in \mathfrak{S}^{s+1}$.

It is known that to define the coboundary operator δ by

$$\delta f(x_0, x_1, \cdots, x_{s+1})=\sum_{i=0}^{s+1}(-1)^i f(x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{s+1}),$$

we obtain

$$H^s(X, \mathfrak{R}) \simeq B^s(X, \mathfrak{R}) / Z^s(X, \mathfrak{R}),$$

if X is normal paracompact (cf. [1], [7]). Here $B^s(X, \mathfrak{R})$ and $Z^s(X, \mathfrak{R})$ are defined as usual for the group of Alexander-Spanier s-cochains (or continuous, regular or alternative s-cochains) and $H^s(X, \mathfrak{R})$ is the Čech cohomology group.

1. Definition of the integral. We use following notations:

$I^s=\{(t_1, \cdots, t_s) \mid 0 \leqq t_1 \leqq 1, \cdots, 0 \leqq t_s \leqq 1\}$,

$J=(j_1, \cdots, j_s)$, $j_1, \cdots, j_s$ are 0 or natural numbers,

$J+1_i=(j_1, \cdots, j_{i-1}, j_i+1, j_{i+1}, \cdots, j_s)$,

$$a_J=(a_{1,j_1},\cdots,a_{s,j_s}),\ 0\leqq a_{1,j_1}\leqq 1,\cdots,0\leqq a_{s,j_s}\leqq 1.$$

Definition. If $f^s=f(x_0,x_1,\cdots,x_s)$ is defined on a neighborhood $U(\Delta_s(X))$ of $\Delta_s(X)$ and $\varphi: I^s\to X$ is a (qubical) singular s-simplex of X ([5]), then we set

$$\int_{\varphi(I^s)} f^s=\lim_{|a_{J+1_i}-a_J|\to 0}\sum_J f(\varphi(a_J),\varphi(a_{J+1_1}),\cdots,\varphi(a_{J+1_s})), \tag{1}$$

if the limit exists. Here $\{a_{i,j_i}\}$ satisfies

$$0=a_{i,0}<a_{i,1}<\cdots<a_{i,n_i}<a_{i,n_i+1}=1.$$

If $\mathfrak{R}$ is a normed vector space, we call f^s is absolutely integrable on $\varphi(I^s)$ if the limit of (1) converges absolutely.

Lemma 1. *The existence or non-existence and the value of* $\int_{\varphi(I^s)} f^s$ *(if it exists) depend only on the germ of* f^s.

Definition. We define the integral $\int_{\varphi(I^s)}\bar{f}^s$ of an Alexander-Spanier s-cochain $\bar{f}^s$ on a singular simplex $\varphi: I^s\to X$ by

$$\int_{\varphi(I^s)}\bar{f}^s=\int_{\varphi(I^s)} f^s, \tag{2}$$

where f^s is a representation of $\bar{f}^s$.

By definition, we get

$$\int_{\varphi(I^s)}(\alpha\bar{f}^s+\beta\bar{g}^s)=\alpha\int_{\varphi(I^s)}\bar{f}^s+\beta\int_{\varphi(I^s)}\bar{g}^s. \tag{3}$$

Lemma 2. *If* f^s *is absolutely integrable on* $\varphi(I^s)$ *and* $\psi(I^s)$, *then*

$$\int_{\varphi(I^s)+\psi(I^s)} f^s=\int_{\varphi(I^s)} f^s+\int_{\psi(I^s)} f^s. \tag{4}$$

Definition. We define the integral $\int_\gamma f^s$ of an Alexander-Spanier cochain f^s on a (qubical) singular s-chain $\gamma=\sum\alpha_i\varphi_i(I^s)$ by

$$\int_\gamma f^s=\sum\alpha_i\int_{\varphi_i(I^s)} f^s, \tag{5}$$

if f^s is absolutely integrable on each $\varphi_i(I^s)$.

By definition, we get

Theorem 1. *If* φ *does not depend on* t_i *and* f^s *satisfies*

$$f^s(x_0,x_1,\cdots,x_s)=0,\qquad \text{if } x_0=x_i,$$

then $\int_{\varphi(I^s)} f^s=0$.

Corollary. *If* f^s *is regular and* $\varphi(I^s)$ *is degenerate, then*

$$\int_{\varphi(I^s)} f^s=0.$$

Theorem 2. *If* f^s *is alternative, then*

$$\int_{\varphi(\sigma(I^s))} f^s=\operatorname{sgn}(\sigma)\int_{\varphi(I^s)} f^s,\qquad \sigma\in\mathfrak{S}^s, \tag{6}$$

where σ *operates on* I^s *by*

$$(t_1,\cdots,t_s)=(t_{\sigma(1)},\cdots,t_{\sigma(s)}).$$

2. Examples. **Theorem 3.** *Setting*

$$f(\varphi(t_1, \cdots, t_s), \varphi(t_1, \cdots, t_s), \cdots, \varphi(t_1, \cdots, t_s))$$
$$= g(t_1, \cdots, t_s, t_{11}, \cdots, t_{1s}, \cdots, t_{s1}, \cdots, t_{ss}),$$
$$0 \leqq t_{ij} \leqq 1, i=1, \cdots, s, j=1, \cdots, S,$$

if g is smooth in each t_{ii}, then

$$\int_{\varphi(I^s)} f^s = \int_{I^s} \frac{\partial^s g}{\partial t_{11} \cdots \partial t_{ss}} \bigg|_{t_{ij}=t_j} dt_1 \cdots dt_s, \tag{7}$$

where the right hand side is the usual Riemannian integral.

Corollary. *If X is a smooth manifold, f^s is a smooth cochain, that is, a representation of f^s is smooth, and is a smooth map, then f^s is absolutely integrable on $\varphi(I^s)$.*

On the other hand, taking $X=I^1$, $f(x_0, x_1)=g(x_0)(x_1-x_0)$, where $g(x)$ is a (Riemannian) integrable function on X, and φ to be the identity map, we get

$$\int_{\varphi(I^1)} f(x_0, x_1) = \int_0^1 g(x)dx,$$

where the right hand side is the Riemannian integral of $g(x)$. Similarly, if we use alternative 1-cochain

$$f(x_0, x_1) = \frac{1}{2}(g(x_0)+g(x_1))(h(x_1)-h(x_0)),$$

and φ is as above, we get

$$\int_{\varphi(I^1)} f(x_0, x_1) = L\int_0^1 g dh,$$

where the right hand side is the Lane-Stieltjes integral ([3], [6]).

On the other hand, if $r(x, y)$ is a metric of X, then we may consider r to be a l-cochain of X. In this case, $\int_{\varphi(I^1)} r$ is the length of (the curve) $\varphi(I^1)$ by this metric.

3. Stokes' theorem. We set the following condition for f^s (and $\varphi(I^s)$) by (*).

(*) *f^s is absolutely integrable on $\varphi(I^s)$ and for any $\varepsilon>0$, there exist $\delta>0$ and $N=N(\varepsilon)>0$ such that*

$$\left\| \int_{\partial(I^s)} f^s - \lim_{|a_{J+1_i}-a_J|\to 0} \sum_{a_J \in I^1_\delta \times \cdots \times I^1_\delta} f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) \right\| < \varepsilon,$$

$$\| f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) \| \leqq N |a_{1,j_1+1}-a_{1,j_1}| \cdots |a_{s,j_s+1}-a_{s,j_s}|,$$

if $a_J \in I^1_\delta \times \cdots \times I^s_\delta$ and each $|a_{i,j_i+1}-a_{i,j_i}|$ is sufficiently small, where each I^k_δ is given by

$$I^k_\delta = \bigcup_{i=0}^{m_k} [b^k{}_{2i}, b^k{}_{2i+1}], \; 0 \leqq b^k{}_0 < b^k{}_1 < \cdots < b^k{}_{2m_k} < b^k{}_{2m_k+1} \leqq 1,$$
$$\sum (b^k{}_{2i+1} - b^k{}_{2i}) > 1-\delta.$$

Theorem 4. *If an $(s+1)$-chain $\gamma = \sum \alpha_i \varphi_i(I^{s+1})$ and an alternative s-cochain f^s satisfies*

(i) *$(\delta f)^{s+1}$ is absolutely integrable and satisfies (*) on each $\varphi_i(I^{s+1})$,*

(ii) *f^s is absolutely integrable and satisfies (*) on each (singular)*

simplex of $\partial\varphi_i(I^{s+1})$,
then we have

(8) $$\int_{\gamma}(\delta f)^{s+1}=\int_{\partial\gamma}f^s.$$

Note. If $s=0$, then we have (8) with no assumption about $f^0=f(x)$.

4. Volume element with respect to a metric. We assume X is an n-dimensional CW complex and fix its CW complex structure. Then we may consider $\int_X f^n$ for an Alexander-Spanier n-cochain f^n of X.

We assume that the topology of X is given by a metric $r=r(x,y)$.

Definition. Setting

(9) $$v(x_0,x_1,\cdots,x_n)=r(x_0,x_1)r(x_0,x_2)\cdots r(x_0,x_n),$$

we call the Alexander-Spanier n-cochain with representation v the volume element of X with respect to the metric r.

Note. Since we can prove that if $X=(U,h_U)$ is an n-dimensional topological manifold with connection t (cf. [2]), then X has a measure m such that

(i) *$h_U^*(m)$ is bi-absolutely continuous with the Lebesgue measure of $\boldsymbol{R}^n$ for all U,*

(ii) *m is invariant under the operation of t,*

if and only if the structure group of the tangent microbundle of X (as an $H_*(n)$-bundle (cf. [2])) is reduced to the group of the germs of all Lebesgue measure preserving homeomorphisms of $\boldsymbol{R}^n$. Hence we obtain

Theorem 5. *The structure group of the tangent microbundle of X (as an $H_*(n)$-bundle) is reduced to the group of the germs of all Lebesgue measure preserving homeomorphisms of $\boldsymbol{R}^n$ if X has an invariant metric under the operation of a connection.*

Next, we assume that X is compact.

In $X\times X$, we denote

$$p_1((x,y))=x,\qquad p_2((x,y))=y.$$

If E and F are vector bundles over X, $k(x,y): p_1^*(E)\to p_2^*(F)$ is a bundle map on $X\times X-\Delta_1(X)$ such that

$$\|k(x,y)\|\leqq M(r(x,y))^{1-n},$$

for some positive M. Then we can define the integral transformation $I(k):\Gamma(E)\to\Gamma(F)$, where $\Gamma(E)$ and $\Gamma(F)$ are the spaces of cross-sections of E and F, by

$$I(k)(f)=\int_X k(x,y)f(x)v(x_0,x_1\cdots,x_n).$$

We denote the space of all bundle maps from $p_1^*(E)$ to $p_2^*(F)$ on $X\times X$ by Hom $(p_1^*(E),p_2^*(F))$, then we define (cf. [4]),

Definition. The class of $k(x,y)$ mod Hom $(p_1^*(E),p_2^*(F))$ is called the symbol of $I(k)$ and denoted by $\sigma(I(k))$.

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