66. Uniformities for Function Spaces and Continuity Conditions

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1. Introduction. Let $\mathfrak{C}$ be, for example, a semi-group of continuous mappings of a uniform space X into itself. For given set-entourage uniformities on $\mathfrak{C}$ a number of properties have been studied. In the other way, we take the following basic criterion on the uniformities for $\mathfrak{C}$ in order to find out new natural uniformities on $\mathfrak{C}$ if possible:

1) the mapping $(u, x)\to u(x)$ of $\mathfrak{C}\times X$ into X is continuous,

2) the mapping $(u, v)\to uv$ of $\mathfrak{C}\times\mathfrak{C}$ into $\mathfrak{C}$ is continuous.

Under this view we get Theorems 2, 3, 4, and 5 in the present paper, and then apply some of the results to get Theorems 6 and 7.

Proofs are omitted, most of which are straightforward. For terms and notations we follow Bourbaki [2].

2. Uniformizability condition.

Theorem 1. *Let X be a set; let Y be a set endowed with a uniform structure $\mathfrak{U}$ which is not the coarsest; let $\mathfrak{S}$ be a family of subsets of X; and let $\mathfrak{F}$ be the family of all mappings of X into Y. For each $A\in\mathfrak{S}$ and each $U\in\mathfrak{U}$, let $W(A, U)$ denote the set of all pairs (u, v) of mappings of X into Y such that $(u(x), v(x))\in U$ for all $x\in A$. Then, as A runs through $\mathfrak{S}$ and U runs through $\mathfrak{U}$, the sets $W(A, U)$ form a fundamental system of entourages of a uniformity on $\mathfrak{F}$ if and only if for any two sets $A_1, A_2\in\mathfrak{S}$ there exists a set $A_3\in\mathfrak{S}$ such that $A_3\supset A_1\cup A_2$.* (1)

Definition 1. For a family $\mathfrak{S}$ that satisfies (1), the uniformity on $\mathfrak{F}$ generated by $\{W(A, U)\,|\,A\in\mathfrak{S}, U\in\mathfrak{U}\}$ is called *$\mathfrak{S}$-uniformity.*

Proposition 1. *Let $X, Y, \mathfrak{U}$ and $\mathfrak{F}$ be the same as those in Theorem 1. Let $\mathfrak{S}$ be a non-empty family of subsets of X; let $\mathfrak{S}^*$ be the family of all sets that are finite unions of sets belonging to $\mathfrak{S}$; let $\mathfrak{S}^{**}$ be the family of all subsets of sets belonging to $\mathfrak{S}^*$; for each $A\in\mathfrak{S}$ let $\mathfrak{W}_A$ denote the $\{A\}$-uniformity on $\mathfrak{F}$, where $\{A\}$ is the family consisting of the set A only. Let $\mathfrak{W}^*$ and $\mathfrak{W}^{**}$ denote $\mathfrak{S}^*$-, and $\mathfrak{S}^{**}$-uniformity on $\mathfrak{F}$ respectively, and let $\mathfrak{W}$ denote the uniformity of $\mathfrak{S}$-convergence in the sense of Bourbaki* [2]. *Then*

$$\mathfrak{W}=\mathfrak{W}^*=\mathfrak{W}^{**}=\bigcup_{A\in\mathfrak{S}^*}\mathfrak{W}_A=\bigcup_{A\in\mathfrak{S}^{**}}\mathfrak{W}_A.$$

The following propositions are the simple cases where some properties of $\mathfrak{W}$ determine $\mathfrak{S}$ and $\mathfrak{U}$.

Proposition 2. *The following conditions are equivalent:*

1) $\mathfrak{W}$ *is the coarsest uniformity on* $\mathfrak{F}$,

2) *the topology on* $\mathfrak{F}$ *induced by* $\mathfrak{W}$ *is the coarsest one,*

3) $\mathfrak{S}$ *consists of the empty set only or* $\mathfrak{U}$ *is the coarsest uniformity on* Y.

Proposition 3. *If the space* Y *contains more than one point, then the following conditions are equivalent:*

1) $\mathfrak{W}$ *is the finest uniformity,*

2) *the topology on* $\mathfrak{F}$ *induced by* $\mathfrak{W}$ *is the finest one,*

3) $X \in \mathfrak{S}^*$ *and* $\mathfrak{U}$ *is the finest uniformity on* Y.

The uniformity $\mathfrak{W}$ depends on both $\mathfrak{S}$ and $\mathfrak{U}$, and in general any uniformity on $\mathfrak{F}$ can not always be defined by giving only $\mathfrak{S}$ adequate properties (cf. Proposition 3), while most uniformities on $\mathfrak{F}$ in the literature have been defined so.

3. Simultaneous continuity.

Notation 1. Let X be a topological space, Y be a uniform space with a uniform structure $\mathfrak{U}$, $\mathfrak{S}$ be a non-empty family of subsets of X, $\mathfrak{S}^*$ be the family of all finite unions of sets belonging to $\mathfrak{S}$, and $\bar{\mathfrak{S}}$ be the family $\{\bar{A} \mid A \in \mathfrak{S}\}$. Let $\mathfrak{F}(X; Y)$ be the family of all mappings of X into Y, $\mathfrak{C}(X; Y)$ be the family of all continuous mappings of X into Y, and $\mathfrak{F}$ (resp. $\mathfrak{C}$) be any non-empty subfamily of $\mathfrak{F}(X; Y)$ (resp. $\mathfrak{C}(X; Y)$). Let $\mathfrak{W}, \bar{\mathfrak{W}}$, and $\mathfrak{W}_c$ be the uniformity of $\mathfrak{S}$-, $\bar{\mathfrak{S}}$-, and compact-convergence on $\mathfrak{F}(X; Y)$ respectively.

Definition 2. A uniformity $\mathfrak{W}$ on $\mathfrak{F}(X; Y)$ is called a uniformity (resp. admissible uniformity) which gives simultaneous continuity for $\mathfrak{C}$ in brief an *s.c.-uniformity* (resp. *a.s.c.-uniformity*) *for* $\mathfrak{C}$, if the mapping $(u, x) \to u(x)$ of $\mathfrak{F}(X; Y) \times X$ (resp. $\mathfrak{C} \times X$) into X is continuous everywhere in $\mathfrak{C} \times X$ with respect to the topology on $\mathfrak{F}(X; Y)$ (resp. the relative topology on $\mathfrak{C}$) induced by $\mathfrak{W}$.

Theorem 2. *Consider the following three conditions:*

1) *every point of* X *is interior to at least one set of* $\mathfrak{S}^*$,

2) $\mathfrak{W}$ *is an s.c.-uniformity for* $\mathfrak{C}$,

3) $\mathfrak{W}$ *is finer than* $\mathfrak{W}_c$.

Then 1)$\Rightarrow$2), 3). *If* $\mathfrak{U}$ *is not the coarsest uniformity,* 2)$\Rightarrow$1), 3). *If* X *is locally compact,* 3)$\Rightarrow$2). *If* X *is locally compact and* $\mathfrak{U}$ *is not the coarsest uniformity,* 3)$\Rightarrow$1).

These are near to the facts in Arens [1] and Dieudonné [3].

Theorem 3. *Consider the following four conditions:*

1) *every point of* X *is interior to at least one set of* $\bar{\mathfrak{S}}^*$,

2) $\bar{\mathfrak{W}}$ *is an s.c.-uniformity for* $\mathfrak{C}(X; Y)$,

3) $\bar{\mathfrak{W}}$ *is an a.s.c.-uniformity for* $\mathfrak{C}(X; Y)$,

4) $\mathfrak{W}$ *is an a.s.c.-uniformity for* $\mathfrak{C}(X; Y)$.

Then 1)$\Rightarrow$2)$\Rightarrow$3)$\Rightarrow$4). *If* $\mathfrak{U}$ *is not the coarsest uniformity,* 2)$\Rightarrow$1). If

X is a completely regular space and Y is a Hausdorff uniform space which contains a non-degenerate arc, 3)$\Rightarrow$1).

Following corollaries hold under the conditions for X and Y in the case where 3)$\Rightarrow$1) of Theorem 3.

Corollary 1. *If $\mathfrak{S}$ consists of closed subsets of X, then $\mathfrak{W}$ is an a.s.c.-uniformity for $\mathfrak{C}(X;Y)$ if and only if it is an s.c.-uniformity for $\mathfrak{C}(X;Y)$.*

Corollary 2. *If $\mathfrak{W}$ is an a.s.c.-uniformity for $\mathfrak{C}(X;Y)$, then $\mathfrak{W}$ is finer than $\mathfrak{W}_c$ on $\mathfrak{C}(X;Y)$.*

4. Continuity of uv with respect to the topology on $\mathfrak{F}(X)$.

Notation 2. Let X be a uniform space with a uniform structure $\mathfrak{U}$, and $\mathfrak{S}, \mathfrak{S}^*, \bar{\mathfrak{S}}$ be the same as in Notation 1. Let $\mathfrak{F}(X)$ (resp. $\mathfrak{C}(X)$) be the family of all mappings (resp. all continuous mappings) of X into itself, and $\mathfrak{F}$ (resp. $\mathfrak{C}$) be any non-empty subfamily of $\mathfrak{F}(X)$ (resp. $\mathfrak{C}(X)$). Let $\mathfrak{W}$ (resp. $\bar{\mathfrak{W}}$) be the uniformity of $\mathfrak{S}$-convergence (resp. $\bar{\mathfrak{S}}$-convergence) on $\mathfrak{F}(X)$. For any two mappings u, v belonging to $\mathfrak{F}(X)$, uv will always be the composite mapping $x \to u(v(x))$ $(x \in X)$.

These notations will keep the meanings hereafter throughout the paper.

Definition 3. A uniformity $\mathfrak{W}$ on $\mathfrak{F}(X)$ is called a *p.-uniformity for* $\mathfrak{F}$, if the mapping $(u, v) \to uv$ is continuous everywhere in $\mathfrak{F} \times \mathfrak{F}$ with respect to the topologies on $\mathfrak{F}(X) \times \mathfrak{F}(X)$ and $\mathfrak{F}(X)$ which are induced by the uniformity $\mathfrak{W}$. Let $\mathfrak{F}'$ be another non-empty subfamily of $\mathfrak{F}(X)$ which is closed under the mapping composition. A uniformity $\mathfrak{W}$ on $\mathfrak{F}(X)$ is called an *a.p.-uniformity for* $\mathfrak{F}'$, if the mapping $(u, v) \to uv$ is continuous everywhere in $\mathfrak{F}' \times \mathfrak{F}'$ with respect to the relative topologies on $\mathfrak{F}' \times \mathfrak{F}'$ and $\mathfrak{F}'$ which are induced by the uniformity $\mathfrak{W}$.

It is evident that a p.-uniformity for $\mathfrak{F}'$ implies an a.p.-uniformity for $\mathfrak{F}'$.

Lemma 1 (Dieudonné [3]). *Let u_0 and v_0 be two fixed mappings belonging to $\mathfrak{F}(X)$. If for an arbitrary set $A \in \mathfrak{S}^*$, there exist a set $B \in \mathfrak{S}^*$ and an entourage $U \in \mathfrak{U}$ such that*

1) $U(v_0(A)) \subset B$, and 2) u_0 is uniformly continuous on B, then the mapping $(u, v) \to uv$ of $\mathfrak{F}(X) \times \mathfrak{F}(X)$ into $\mathfrak{F}(X)$ is continuous at (u_0, v_0) with respect to the topologies induced by the uniformity $\mathfrak{W}$.

Theorem 4. *Let $\mathfrak{F}$ be any non-empty subfamily of $\mathfrak{F}(X)$. If*

1) for each mapping $u \in \mathfrak{F}$ and each set $A \in \mathfrak{S}^*$, there exists a set $B \in \mathfrak{S}^*$ such that $u(A) \subset B$,

2) for each set $A \in \mathfrak{S}^*$, there exist an entourage $U \in \mathfrak{U}$ and a set $C \in \mathfrak{S}^*$ such that $U(A) \subset C$, and

3) each mapping $u \in \mathfrak{F}$ is uniformly continuous on every set $A \in \mathfrak{S}^*$, then the uniformity $\mathfrak{W}$ is a p.-uniformity for $\mathfrak{F}$. The converse is also true if $\mathfrak{F}$ contains the identity mapping and $\mathfrak{U}$ is not the coarsest uniformity.

Remarks. i) If the identity mapping of X is contained in $\mathfrak{F}$, the conjunction of conditions 1) and 2) in Theorem 4 is equivalent to that the condition 1) in Lemma 1 holds for any $v_0 \in \mathfrak{F}$. ii) If the condition

4) X is covered by $\mathfrak{S}^*$,

is combined with the conditions 2) and 3) in Theorem 4, it is easily seen that

a) the condition 2) implies that $\bar{\mathfrak{W}}$ coincides with $\bar{\bar{\mathfrak{W}}}$,

b) the conditions 2) and 4) imply that every point of X is interior to at least one set of $\mathfrak{S}^*$, and

c) conditions 2), 3) and 4) imply that $\mathfrak{F} \subset \mathfrak{C}(X)$.

5. Continuity of *uv* with respect to the relative topology on $\mathfrak{C}$. To consider a.p.-uniformity, we take the following weaker conditions 1)′, 2)′ and 3)′ than those in Theorem 4.

Lemma 2. *Let $\mathfrak{C}$ be any non-empty subfamily of $\mathfrak{C}(X)$ which is closed under the mapping composition. If*

1)′ *for each mapping $u \in \mathfrak{C}$ and each set $A \in \mathfrak{S}^*$, there exists a set $B \in \mathfrak{S}^*$ such that $u(A) \subset \bar{B}$,*

2)′ *for each set $A \in \mathfrak{S}^*$, there exist an entourage $U \in \mathfrak{U}$ and a set $C \in \mathfrak{S}^*$ such that $U(A) \subset \bar{C}$, and*

3)′ *every mapping $u \in \mathfrak{C}$ is uniformly continuous on $\bar{A}$ for every set $A \in \mathfrak{S}^*$,*

then the uniformity $\mathfrak{W}$ is an a.p.-uniformity for $\mathfrak{C}$. Conversely, in the case where $\mathfrak{C}$ is $\mathfrak{C}(X)$ and X is a Hausdorff uniform space that contains a non-degenerate arc, the condition 1)′ holds if $\mathfrak{W}$ is an a.p.-uniformity for $\mathfrak{C}(X)$.

For the proof of the remaining two conditions in Lemma 2 from a.p.-uniformity, we need an auxiliary concept "uniform deformability" as follows.

Definition 4. A uniform space X with a uniformity $\mathfrak{U}$ is *uniformly deformable* if for any entourage $U \in \mathfrak{U}$ there exists an entourage $V \in \mathfrak{U}$ as follows: for any two V-close points p and q, there exists a continuous mapping f of X into itself such that $f(p)=q$ and $(x, f(x)) \in U$ for any $x \in X$.

Ford [4] defined the similar notion "strong local homogeneity" which is stronger than ours in those points that f must be a homeomorphism and fixes the complement of a neighborhood of x, while weaker than ours in the point that the uniform scale of such neighborhoods is not required. There are several examples common to his and ours.

Lemma 3. *Let X be a uniformly deformable Hausdorff space that contains a non-degenerate arc. If $\mathfrak{W}$ is an a.p.-uniformity for $\mathfrak{C}(X)$, then conditions 2)′ and 3)′ in Lemma 2 hold for $\mathfrak{C}=\mathfrak{C}(X)$.*

Theorem 5. *Let X be a uniformly deformable Hausdorff space with a uniform structure $\mathfrak{U}$ which contains a non-degenerate arc, and $\mathfrak{C}(X)$ be the family of all continuous mappings of X into itself. Then $\mathfrak{W}$ is an a.p.-uniformity for $\mathfrak{C}(X)$ if and only if the following three conditions hold:*

1) *for each mapping $u \in \mathfrak{C}(X)$ and each set $A \in \mathfrak{S}^*$, there exists a set $B \in \mathfrak{S}^*$ such that $u(A) \subset \bar{B}$,*

2) *for each set $A \in \mathfrak{S}^*$, there exist an entourage $U \in \mathfrak{U}$ and a set $C \in \mathfrak{S}^*$ such that $U(A) \subset \bar{C}$, and*

3) *every mapping $u \in \mathfrak{C}(X)$ is uniformly continuous on $\bar{A}$ for any set $A \in \mathfrak{S}^*$.*

Among spaces to which Theorem 5 can be applied,

i) locally euclidean, uniformly locally connected, uniform spaces,

ii) locally euclidean, compact, uniform spaces, and

iii) convex subsets of any normed linear space

are important. (Note that our results thus far obtained remain true if the word "Hausdorff" is replaced by "locally Hausdorff".)

The following corollaries hold under the same condition on X as in Theorem 5.

Corollary 1. *$\mathfrak{W}$ is an a.p.-uniformity for $\mathfrak{C}(X)$ if and only if $\overline{\mathfrak{W}}$ is a p.-uniformity for $\mathfrak{C}(X)$.*

Corollary 2. *If all sets in $\mathfrak{S}$ are closed, then $\mathfrak{W}$ is an a.p.-uniformity for $\mathfrak{C}(X)$ if and only if it is a p.-uniformity for $\mathfrak{C}(X)$.*

6. Uniqueness of set-entourage uniformities. Using 2) and 3) in Theorem 5 and the structure of *UC* spaces (cf. [6] and [7]), we have Theorems 6 and 7 below concerning a kind of uniqueness of uniformity.

Definition 5. A topological space is *locally dense* if there exists a base consisting of dense-in-itself open sets.

Theorem 6 (Karube [5]). *Let X be a uniformly deformable, locally dense, metric space that contains a non-degenerate arc. If $\mathfrak{W}$ is an a.p.-uniformity and an a.s.c.-uniformity for $\mathfrak{C}(X)$, then $\mathfrak{W}$ coincides with $\mathfrak{W}_c$ on $\mathfrak{C}(X)$. The converse is true if X is locally compact uniform space.*

Lemma 4. *Let X be a uniformly deformable Hausdorff space that contains a non-degenerate arc. If $\mathfrak{W}$ is a Hausdorff a.p.-uniformity for $\mathfrak{C}(X)$, then $\mathfrak{W}$ is an s.c.-uniformity for $\mathfrak{C}(X)$ and $\mathfrak{W}$ is finer than $\mathfrak{W}_c$.*

Theorem 7. *Let X be a uniformly deformable, locally dense, metric space that contains a non-degenerate arc. If $\mathfrak{W}$ is a Hausdorff a.p.-uniformity for $\mathfrak{C}(X)$, then $\mathfrak{W}$ coincides with $\mathfrak{W}_c$ on $\mathfrak{C}(X)$.*

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