61. An Extension of an Integral. II

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1. Lemmas. This section is the continuation of section 3 in [1].

Assumption 3. *$\mathcal{J}$ is an abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$.*

For each $f \in \mathcal{F}$, we can define a map μ_f of $\mathcal{R}(f)$ into J by $\mu_f(X) = \mathcal{J}(X, Xf)$ for $X \in \mathcal{R}(f)$.

Lemma 13. *The map μ_f is a J-valued pre-measure on $\mathcal{R}(f)$ for any $f \in \mathcal{F}$.*

Lemma 14. *If $f, g \in \mathcal{F}$ and $X \in \mathcal{R}(f) \cap \mathcal{R}(g)$, then $X \in \mathcal{R}(f+g)$ and $\mu_{f+g}(X) = \mu_f(X) + \mu_g(X)$.*

Lemma 15. *Suppose that $f \in \mathcal{F}$, $X \in \mathcal{S}$, and $Y \in \bar{\Sigma}$. Then $XY \in \mathcal{R}(f)$ if and only if $X \in \mathcal{R}(Yf)$, and these mutually equivalent conditions imply that $\mu_f(XY) = \mu_{Yf}(X)$.*

Proof. This follows from Lemma 7 in [1].

Let $\mathcal{V}$ be the system of neighbourhoods of $0 \in J$. Denote by Ω the set of all elements $(X, f) \in \tilde{\Omega}$ satisfying the following condition: for any $\xi, \eta \in \Sigma(f)$ such that $\bar{\xi} = \bar{\eta} = X$ and for any $V \in \mathcal{V}$, there exists a positive integer n such that $\mu_f(\xi(l)) - \mu_f(\eta(m)) \in V$ for any $l \geqq n$ and $m \geqq n$.

Lemma 16. *$(XY, f) \in \Omega$ if and only if $(X, Yf) \in \Omega$ for any $X, Y \in \bar{\Sigma}$ and $f \in \mathcal{F}$.*

Proof. Suppose that $(XY, f) \in \Omega$. Lemma 11 implies that $(X, Yf) \in \tilde{\Omega}$. Let ξ and η be elements of $\Sigma(Yf)$ such that $\bar{\xi} = \bar{\eta} = X$ and let V be an element of $\mathcal{V}$. It follows from Corollary to Lemma 7 that $Y\xi, Y\eta \in \Sigma(f)$ and $\overline{Y\xi} = \overline{Y\eta} = XY$. Hence we have an n such that $\mu_f((Y\xi)(l)) - \mu_f((Y\eta)(m)) \in V$ for any $l \geqq n$ and $m \geqq n$. For this n and for $l \geqq n$ and $m \geqq n$ we have $\mu_{Yf}(\xi(l)) - \mu_{Yf}(\eta(m)) = \mu_f(\xi(l)Y) - \mu_f(\eta(m)Y) = \mu_f((Y\xi)(l)) - \mu_f((Y\eta)(m)) \in V$. Thus we have $(X, Yf) \in \Omega$. Conversely suppose that $(X, Yf) \in \Omega$. $(XY, f) \in \tilde{\Omega}$ follows from Lemma 11. Let ζ_i be elements of $\Sigma(f)$ such that $\bar{\zeta}_i = XY$ for $i = 1, 2$, and let V be an element of $\mathcal{V}$. Lemma 8 implies that there are $\xi_i \in \Sigma(Yf)$ such that $\bar{\xi}_i = X$ and $\zeta_i = Y\xi_i$ for $i = 1, 2$. Since $(X, Yf) \in \Omega$, we have an n such that $\mu_{Yf}(\xi_1(l_1)) - \mu_{Yf}(\xi_2(l_2)) \in V$ for any $l_i \geqq n$. For this n and for $l_i \geqq n$, $i = 1, 2$, we have $\mu_f(\zeta_1(l_1)) - \mu_f(\zeta_2(l_2)) = \mu_f((Y\xi_1)(l_1)) - \mu_f((Y\xi_2)(l_2)) = \mu_f(\xi_1(l_1)Y) - \mu_f(\xi_2(l_2)Y) = \mu_{Yf}(\xi_1(l_1)) - \mu_{Yf}(\xi_2(l_2)) \in V$, which implies that $(XY, f) \in \Omega$. Thus the lemma is proved.

Denote by $\mathcal{S}(f)$ the set $\{X|(X, f) \in \Omega\}$ for each $f \in \mathcal{F}$. We have another expression of $\mathcal{S}(f)$ as follows:

Lemma 17. *For any $f \in \mathcal{F}$, $\mathcal{S}(f)$ is the set of all elements $X \in \overline{\Sigma(f)}$ satisfying the following condition: for any $\xi, \eta \in \Sigma(f)$ such that $\bar{\xi}=\bar{\eta}=X$ and for any $V \in \mathcal{V}$, there exists a positive integer n such that $\mu_f(\xi(l))-\mu_f(\eta(m)) \in V$ for any $l \geqq n$ and $m \geqq n$.*

Lemma 18. *For any $f \in \mathcal{F}$, $\mathcal{S}(f)$ is an ideal of $\bar{\Sigma}$ and is a pseudo-σ-ring.*

Proof. It is sufficient to show that 1) $XY \in \mathcal{S}(f)$ for any $X \in \mathcal{S}(f)$ and $Y \in \bar{\Sigma}$, and 2) $X_1+X_2 \in \mathcal{S}(f)$ for any $X_1, X_2 \in \mathcal{S}(f)$ such that $X_1X_2=0$. Let us first prove 1). Put $Z=XY$. $X \in \mathcal{S}(f)$ implies that $X \in \overline{\Sigma(f)}$ and hence it follows from Lemma 6 that $Z=XY \in \overline{\Sigma(f)}$. Let ξ and η be elements of $\Sigma(f)$ such that $\bar{\xi}=\bar{\eta}=Z$ and let V be an element of $\mathcal{V}$. Assume that for any positive integer n there were $l_n \geqq n$ and $m_n \geqq n$ such that $\mu_f(\xi(l_n))-\mu_f(\eta(m_n)) \notin V$. It follows from our assumption that there are sequences l_k and m_k, $k=1,2,\cdots$, such that $\max(l_k, m_k) < \min(l_{k+1}, m_{k+1})$ and such that $\mu_f(\xi(l_k))-\mu_f(\eta(m_k)) \notin V$ for each $k=1,2,\cdots$. Now Lemma 6 implies that $X+Z \in \overline{\Sigma(f)}$ and hence we can write $X+Z=\bar{\zeta}$ for some $\zeta \in \Sigma(f)$. Putting $\xi'(n)=\xi(l_n)+\zeta(n)$ and $\eta'(n)=\eta(m_n)+\zeta(n)$ for $n=1,2,\cdots$, we have $\xi', \eta' \in \Sigma(f)$, $\bar{\xi}'=\bar{\xi}+\bar{\zeta}=Z+(X+Z)=X$, and $\bar{\eta}'=X$. Since $X \in \mathcal{S}(f)$ we have an n such that $\mu_f(\xi'(l))-\mu_f(\eta'(m)) \in V$ for any $l \geqq n$ and $m \geqq n$. On the other hand we have $\mu_f(\xi'(n))-\mu_f(\eta'(n))=\mu_f(\xi(l_n)+\zeta(n))-\mu_f(\eta(m_n)+\zeta(n))=\mu_f(\xi(l_n))-\mu_f(\eta(m_n)) \notin V$, which is a contradiction. Hence we have an n such that $\mu_f(\xi(l))-\mu_f(\eta(m)) \in V$ for any $l \geqq n$ and $m \geqq n$, and thus Lemma 17 implies that $XY=Z \in \mathcal{S}(f)$.

Now let us prove 2). That $X_1+X_2 \in \overline{\Sigma(f)}$ follows from Lemma 6. Let ξ and η be elements of $\Sigma(f)$ such that $\bar{\xi}=\bar{\eta}=X_1+X_2$ and let V be an element of $\mathcal{V}$. We have $U \in \mathcal{V}$ such that $2U \subset V$. Since $X_i\xi$ and $X_i\eta$ are elements of $\Sigma(f)$ (Corollary 2 to Lemma 6), since $\overline{X_i\xi}=X_i\bar{\xi}=X_i(X_1+X_2)=X_i$, and since $\overline{X_i\eta}=X_i$, we have n_i, $i=1,2$, such that $\mu_f(X_i\xi(l_i))-\mu_f(X_i\eta(m_i)) \in U$ for any $l_i \geqq n_i$ and $m_i \geqq n_i$. For $n=\max\cdot(n_1, n_2)$, and for any $l \geqq n$, and any $m \geqq n$, we have $\mu_f(\xi(l))-\mu_f(\eta(m)) = \mu_f((X_1+X_2)\xi(l))-\mu_f((X_1+X_2)\eta(m)) = \{\mu_f(X_1\xi(l))-\mu_f(X_1\eta(m))\} + \{\mu_f(X_2\xi(l))-\mu_f(X_2\eta(m))\} \in U+U \subset V$. Hence it follows that $X_1+X_2 \in \mathcal{S}(f)$ and thus the lemma is proved.

Assumption 4. *For $X_i \in \mathcal{S}$, $i=1,2,\cdots$, such that $X_i \downarrow 0$ $(i \to \infty)$, and for any $g=\mathcal{G}$, it holds that $\mathcal{J}(X_i, g) \to 0$ $(i \to \infty)$.*

Lemma 19. *The map μ_f is a J-valued measure on $\mathcal{R}(f)$ for any $f \in \mathcal{F}$.*

Proof. Suppose that $X_i \in \mathcal{R}(f)$, $i=1,2,\cdots$, and that $X_i \downarrow 0$ $(i \to \infty)$. Then it follows from Assumption 4 that $\mu_f(X_i)=\mathcal{J}(X_i, X_if)$

$=\mathcal{I}(X_i, X_iX_1f)=\mathcal{I}(X_i, X_1f)\to 0$ $(i\to\infty)$. Hence Lemma 13 implies that μ_f is a measure.

Lemma 20. *$\mathcal{R}(f)\subset\mathcal{S}(f)\subset\overline{\Sigma(f)}\subset\bar{\Sigma}$ for any $f\in\mathcal{F}$.*

Proof. Let us prove that $\mathcal{R}(f)\subset\mathcal{S}(f)$. Let X be an element of $\mathcal{R}(f)$. That $X\in\overline{\Sigma(f)}$ follows from Corollary 1 to Lemma 6. Let ξ_i be elements of $\Sigma(f)$ such that $\bar{\xi}_i=X$, for $i=1,2$, and let V be an element of $\mathcal{V}$. We have $U\in\mathcal{V}$ such that $U-U\subset V$. Since $\xi_i(j)\uparrow X$ $(j\to\infty)$ and since μ_f is a measure, it holds that $\mu_f(\xi_i(j))\to\mu_f(X)$ $(j\to\infty)$. Hence, for $i=1,2$, we have n_i such that $\mu_f(\xi_i(j))-\mu_f(X)\in U$ for any $j\geqq n_i$. For $n=\max(n_1, n_2)$, and for any $l\geqq n$ and any $m\geqq n$, we have $\mu_f(\xi_1(l))-\mu_f(\xi_2(m))=\{\mu_f(\xi_1(l))-\mu_f(X)\}-\{\mu_f(\xi_2(m))-\mu_f(X)\}\in U-U\subset V$. This implies that $X\in\mathcal{S}(f)$ and hence $\mathcal{R}(f)\subset\mathcal{S}(f)$.

Corollary 1. *$\mathcal{S}\subset\mathcal{S}(g)$ for any $g\in\mathcal{G}$.*

Proof. This follows from Lemma 3.

Corollary 2. *$\mathcal{S}\times\mathcal{G}\subset\Omega\subset\tilde{\Omega}\subset\bar{\Sigma}\times\mathcal{F}$.*

Proof. For $(X, g)\in\mathcal{S}\times\mathcal{G}$, Corollary 1 implies that $X\in\mathcal{S}(g)$. This implies $(X, g)\in\Omega$, and hence $\mathcal{S}\times\mathcal{G}\subset\Omega$.

Put $\mathcal{G}(X)=\{f\mid(X, f)\in\Omega\}$ for each $X\in\bar{\Sigma}$. Then we have

Lemma 21. *$\mathcal{G}(X)\subset\tilde{\mathcal{G}}(X)\subset\mathcal{F}$ for any $X\in\bar{\Sigma}$. Further $\mathcal{G}\subset\mathcal{G}(X)$ if $X\in\mathcal{S}$.*

Proof. For $X\in\mathcal{S}$, $\mathcal{G}\subset\mathcal{G}(X)$ follows from Corollary 2 to Lemma 20.

Lemma 22. *Suppose that $X\in\bar{\Sigma}, f_i\in\mathcal{G}(X)$ for $i=1,2,\cdots,n$, and that $f_0\in\tilde{\mathcal{G}}(X)$. Further suppose for any $V\in\mathcal{V}$ there exists $U\in\mathcal{V}$ satisfying the following condition: if Y and Z are elements of $\bigcap_{i=0}^{n}\mathcal{R}(f_i)$ such that $Y\subset X$ and $Z\subset X$, and if $\mu_{f_i}(Y)-\mu_{f_i}(Z)\in U$ for any $i=1,2,\cdots,n$, then $\mu_{f_0}(Y)-\mu_{f_0}(Z)\in V$. Then it holds that $f_0\in\mathcal{G}(X)$.*

Proof. We are proving that $(X, f_0)\in\Omega$. Let ξ and η be elements of $\Sigma(f_0)$ such that $\bar{\xi}=\bar{\eta}=X$ and let W be an element of $\mathcal{V}$. Write $\mu_i=\mu_{f_i}$ for $i=0,1,\cdots,n$. Then it suffices to show the existence of a positive integer r such that $\mu_0(\xi)p))-\mu_0(\eta(q))\in W$ for any $p\geqq r$ and $q\geqq r$. Let us show this.

Since $(X, f_i)\in\tilde{\Omega}$ for any $i=0,1,\cdots,n$, Corollary 2 to Lemma 10 implies the existence of $\zeta\in\bigcap_{i=0}^{n}\Sigma(f_i)$ such that $\bar{\zeta}=X$. Put $\mathcal{N}=\{(j, k)\mid j$ and k are positive integers$\}$ and write $(j, k)\leqq(j', k')$, for $(j, k), (j', k')\in\mathcal{N}$, if and only if the two inequalities $j\leqq j'$ and $k\leqq k'$ hold. Then $\mathcal{N}$ becomes a directed set and hence, putting $a_{(j,k)}=\mu_0(\xi(j)\zeta(k))$, we have a directed sequence $a_{(j,k)}, (j, k)\in\mathcal{N}$, in J.

We assert that the sequence $a_{(j,k)}, (j, k)\in\mathcal{N}$, is a Cauchy sequence. Suppose this were false. Then we have an element V_0 of $\mathcal{V}$ satisfying the condition: for any $(j, k)\in\mathcal{N}$ there is $(j', k')\in\mathcal{N}$ such that $(j, k)\leqq(j', k')$ and $a_{(j',k')}-a_{(j,k)}\notin V_0$. Thus we have sequences of positive integers j_m, k_m, and l_m, where $m=1,2,\cdots$, such that, for each m, l_{m+1}

$=\max(m, j_m, k_m)$, $(l_m, l_m)\leqq(j_m, k_m)$, and $a_{(j_m,k_m)}-a_{(l_m,l_m)}\notin V_0$. Then $(l_1, l_1)\leqq(j_1, k_1)\leqq(l_2, l_2)\leqq(j_2, k_2)\leqq\cdots$, and $\lim_{m\to\infty} l_m=\infty$. Put $\lambda(2m-1)=\xi(l_m)\zeta(l_m)$ and $\lambda(2m)=\xi(j_m)\zeta(k_m)$ for $m=1,2,\cdots$. Then we have $\lambda\in\bigcap_{i=0}^n \Sigma(f_i)$ and $\bar{\lambda}=X$. Now let U_0 be an element of $\mathcal{V}$ satisfying the condition: if Y and Z are elements of $\bigcap_{i=0}^n \mathcal{R}(f_i)$ such that $Y\subset X$ and $Z\subset X$, and if $\mu_i(Y)-\mu_i(Z)\in U_0$ for any $i\geqq 1$, then $\mu_0(Y)-\mu_0(Z)\in V_0$. Since $(X, f_i)\in\Omega$ and $\lambda\in\Sigma(f_i)$ for $i\geqq 1$, and since $\bar{\lambda}=X$, we have a positive integer m_i, for each $i=1,2,\cdots,n$, such that $\mu_i(\lambda(p))-\mu_i(\lambda(q))\in U_0$ for any $p\geqq m_i$ and $q\geqq m_i$. Put $m=\max(m_1, m_2,\cdots, m_n)$. Then it follows from $2m>2m-1\geqq m=\max m_i$ that $\mu_i(\lambda(2m))-\mu_i(\lambda(2m-1))\in U_0$ for each $i\geqq 1$. For this m it follows that $a_{(j_m,k_m)}-a_{(l_m,l_m)}=\mu_0(\xi(j_m)\zeta(k_m))-\mu_0(\xi(l_m)\zeta(l_m))=\mu_0(\lambda(2m))-\mu_0(\lambda(2m-1))\in V_0$. This is a contradiction and hence our assertion is true.

That which is proved above implies that, for $W_0\in\mathcal{V}$ such that $-W_0=W_0$ and $8W_0\subset W$, there is $(j_0, k_0)\in\mathfrak{N}$ such that $a_{(j,k)}-a_{(j_0,k_0)}\in W_0$ for any $(j,k)\geqq(j_0,k_0)$. Let U be an element of $\mathcal{V}$ satisfying the condition: if Y and Z are elements of $\bigcap_{i=0}^n \mathcal{R}(f_i)$ such that $Y\subset X$ and $Z\subset X$, and if $\mu_i(Y)-\mu_i(Z)\in U$ for any $i\geqq 1$, then $\mu_0(Y)-\mu_0(Z)\in W_0$. Since $(X, f_i)\in\Omega$, for $i\geqq 1$, since $\xi\zeta$ and ζ are elements of $\Sigma(f_i)$, and since $\overline{\xi\zeta}=\bar{\zeta}=X$, we have a positive integer m_i, for each $i=1,2,\cdots,n$, such that $\mu_i((\xi\zeta)(p))-\mu_i(\zeta(q))\in U$ for any $p\geqq m_i$ and $q\geqq m_i$. Put $r_1=\max(j_0, k_0, m_1, m_2,\cdots, m_n)$.

For the integer r_1 defined above, we shall show that $\mu_0(\xi(p))-\mu_0(\zeta(q))\in 4W_0$ for any $p\geqq r_1$ and $q\geqq r_1$. Since $\mu_i((\xi\zeta)(p))-\mu_i(\zeta(q))\in U$ for any $i\geqq 1$, it follows from the definition of U that $\mu_0((\xi\zeta)(p))-\mu_0(\zeta(q))\in W_0$. Hence, $\mu_0((\xi\zeta(p))=\mu_0(\xi(p)\zeta(p))=a_{(p,p)}$ implies that 1) $a_{(p,p)}-\mu_0(\zeta(q))\in W_0$. Now put $Y=\xi(p)$ and $Y_k=\xi(p)\zeta(k)$ for $k=1,2,\cdots$. Then we have $Y, Y_k\in\mathcal{R}(f_0)$, for each k, and $Y_k\uparrow Y$ $(k\to\infty)$. Hence it follows from Lemma 19 that $\mu_0(Y_k)\to\mu_0(Y)$ $(k\to\infty)$ and thus we have $k_1\geqq k_0$ such that $\mu_0(Y_{k_1})-\mu_0(Y)\in W_0$. For this k_1, $a_{(p,k_1)}=\mu_0(\xi(p)\zeta(k_1))=\mu_0(Y_{k_1})$ implies that 2) $\mu_0(\xi(p))-a_{(p,k_1)}\in W_0$. Further, since $(p, k_1)\geqq(j_0, k_0)$ and since $(p,p)\geqq(j_0,k_0)$, $a_{(p,k_1)}-a_{(j_0,k_0)}$ and $a_{(p,p)}-a_{(j_0,k_0)}$ are elements of W_0 and thus we have 3) $a_{(p,k_1)}-a_{(p,p)}\in 2W_0$. Then it follows from 1), 2), and 3), that $\mu_0(\xi(p))-\mu_0(\zeta(q))\in 4W_0$.

In an analogous way, we have a positive integer r_2 such that $\mu_0(\eta(p))-\mu_0(\zeta(q))\in 4W_0$ for any $p\geqq r_2$ and $q\geqq r_2$. For $r=\max(r_1, r_2)$, and for any $p\geqq r$ and $q\geqq r$, we have $\mu_0(\xi(p))-\mu_0(\eta(q))=\{\mu_0(\xi(p))-\mu_0(\zeta(r))\}-\{\mu_0(\eta(q))-\mu_0(\zeta(r))\}\in 4W_0-4W_0=8W_0\subset W$. This completes the proof of Lemma 22.

Corollary. *For any $X\in\bar{\Sigma}$, $\mathcal{G}(X)$ is a subgroup of $\mathcal{F}$.*

Proof. It suffices to show that $f_1-f_2\in\mathcal{G}(X)$ for given $f_i\in\mathcal{G}(X)$, $i=1,2$. For $f_0=f_1-f_2$, it follows from Lemma 12 that $f_0\in\tilde{\mathcal{G}}(X)$. For any $V \in \mathcal{V}$, there exists $U \in \mathcal{V}$ such that $U - U \subset V$. Let Y and Z be elements of $\bigcap_{i=0}^{2} \mathcal{R}(f_i)$ such that $Y \subset X$ and $Z \subset X$ and suppose that $\mu_{f_i}(Y) - \mu_{f_i}(Z) \in U$ for $i=1,2$. Then $\mu_{f_0}(Y) - \mu_{f_0}(Y) - \mu_{f_0}(Z) = \{\mu_{f_1}(Y) - \mu_{f_2}(Y)\} - \{\mu_{f_1}(Z) - \mu_{f_2}(Z)\} = \{\mu_{f_1}(Y) - \mu_{f_1}(Z)\} - \{\mu_{f_2}(Y) - \mu_{f_2}(Z)\} \in U - U \subset V$. Thus the lemma implies that $f_1 - f_2 = f_0 \in \mathcal{G}(X)$.

Assumption 5. *J is Hausdorff and complete.*

Lemma 23. *For each $f \in \mathcal{F}$, the measure μ_f is uniquely extended to a J-valued measure on $\mathcal{S}(f)$.*

Proof. Lemmas 4, 5, and 17 imply that our lemma follows from Theorems 1 and 2 in [5].

For each $f \in \mathcal{F}$, denote by $\bar{\mu}_f$ the extended measure on $\mathcal{S}(f)$ stated in Lemma 23. Then we have

Lemma 24. *There exists a unique map $\bar{\mathcal{J}}$ of Ω into J satisfying the following condition: if $(X, f) \in \Omega$, if $X_i \in \mathcal{S}$ with $X_i f \in \mathcal{G}$, $i=1,2,\cdots$, and if $X_i \uparrow X$ $(i \to \infty)$, then $\mathcal{J}(X_i, X_i f) \to \bar{\mathcal{J}}(X, f)$ $(i \to \infty)$. Further it holds that $\bar{\mathcal{J}}(X, f) = \bar{\mu}_f(X)$ for any $(X, f) \in \Omega$.*

For the map $\bar{\mathcal{J}}$ of Ω into J stated above we have

Lemma 25. *The map $\bar{\mathcal{J}}$ has the following properties:*

1) *$\bar{\mathcal{J}}$ is an extension of $\mathcal{J}$.*

2) *Suppose that $X, Y \in \bar{\Sigma}$ and that $f \in \mathcal{F}$. Then $(XY, f) \in \Omega$ if and only if $(X, Yf) \in \Omega$. Further these mutually equivalent conditions imply that $\bar{\mathcal{J}}(XY, f) = \bar{\mathcal{J}}(X, Yf)$.*

3) *For any fixed $f \in \mathcal{F}$, the map $\bar{\mathcal{J}}_f(X) = \bar{\mathcal{J}}(X, f)$ on $\mathcal{S}(f)$ is a measure.*

4) *For any fixed $X \in \bar{\Sigma}$, the map $\bar{\mathcal{J}}_X(f) = \mathcal{J}(X, f)$ on $\mathcal{G}(X)$ is a homomorphism.*

Proof. 1) and 3) follow immediately from Lemma 24. Let us prove 2). The equation $\bar{\mathcal{J}}(XY, f) = \bar{\mathcal{J}}(X, Yf)$ is proved as follows. Since $(X, Yf) \in \Omega$, we have $\xi \in \Sigma(Yf)$ such that $\bar{\xi} = X$. Lemma 15 implies that $\mu_{Yf}(\xi(n)) = \mu_f(Y\xi(n))$, for $n=1,2,\cdots$, and hence we have $\bar{\mathcal{J}}(X, Yf) = \bar{\mu}_{Yf}(X) = \lim_{n \to \infty} \mu_{Yf}(\xi(n)) = \lim_{n \to \infty} \mu_f(Y\xi(n)) = \bar{\mu}_f(XY) = \bar{\mathcal{J}}(XY, f)$. To prove 4), suppose that $X \in \bar{\Sigma}$ and that $f, g \in \mathcal{G}(X)$. Then we are proving that $\bar{\mathcal{J}}(X, f+g) = \bar{\mathcal{J}}(X, f) + \bar{\mathcal{J}}(X, g)$. Since $(X, f), (X, g)$, and $(X, f+g)$ are elements of Ω, there exists an element ξ of $\Sigma(f) \cap \Sigma(g) \cap \Sigma(f+g)$ such that $\bar{\xi} = X$. Then it follows that $\bar{\mathcal{J}}(X, f+g) = \bar{\mu}_{f+g}(X) = \lim_{n \to \infty} \mu_{f+g}(\xi(n)) = \lim_{n \to \infty} \{\mu_f(\xi(n)) + \mu_g(\xi(n))\} = \lim_{n \to \infty} \mu_f(\xi(n)) + \lim_{n \to \infty} \mu_g(\xi(n)) = \bar{\mu}_f(X) + \bar{\mu}_g(X) = \bar{\mathcal{J}}(X, f) + \bar{\mathcal{J}}(X, g)$. Thus the lemma is proved.

2. Proof of Theorems 1 and 2 in [1]. Under the notations and the assumptions in section 2 in [1], Assumptions 1 and 2 in section 3 in [1] are satisfied (M is the base space of Γ). Note that $\bar{\mathcal{S}}$ is the σ-ring $\bar{\Sigma}$ in section 3 in [1]. For an element μ of Q, denote by $\mathcal{J} = \mathcal{J}_\mu$

the derived abstract integral from σ relative to μ. Then Assumptions 3, 4, and 5 in section 1 are satisfied. Putting $\Omega_\mu=\{(X,f)\mid(X,f,\mu)\in\Omega\}$, where Ω is the carrier of Γ, we see that Ω_μ coincides with the set Ω in section 1. Denote by $\bar{\mathcal{J}}_\mu$ the map $\bar{\mathcal{J}}$ of Ω_μ into J stated in Lemma 24.

Then, 1), 2), 3), and 4) in Theorem 1 follows from Corollary 2 to Lemma 20, Lemma 16, Lemma 18, and Corollary to Lemma 22, respectively.

To prove Theorem 2, put $\bar{\sigma}(X,f,\mu)=\bar{\mathcal{J}}_\mu(X,f)$ for $(X,f,\mu)\in\Omega$. Then we have a map $\bar{\sigma}$ of Ω into J and it follows from Lemma 25 that $\bar{\sigma}$ satisfies the conditions in Theorem 2. The uniqueness of $\bar{\sigma}$ follows from (i), (ii), and (iv) in the proof of Proposition 1 in [1].

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