# 108. On the Asymptotic Behaviors of Solutions of Difference Equations. II

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As for the applications of Lyapunov functions to the stability problems of difference equations with discrete variable, we can find some results in [2, 3, 5], and [4] concerning the criteria of Popov type for the absolute stability. In this paper, we shall show some other results including the construction of Lyapunov functions, that is, the so-called converse theorems, and the applications to perturbed systems.

The following is a result to show the existence of Lyapunov functions for linear systems, which will be often used to discuss the stability problems for perturbed systems.

**Theorem 1.** *Suppose that $A(t)$ be an $n\times n$ matrix defined for $t\in I_\infty$, and the trivial solution of*

(1) $$x(t+1)=A(t)x(t), \quad x(t_0)=x_0, \quad t\geqq t_0$$

*is generalized exponentially asymptotically stable, where $I_\infty$ is a set of nonnegative integers and $t_0\in I_\infty$. Then there exists a function $V(t,x)$ satisfying the following conditions:*

(a) *$V(t,x)$ is defined for $t\in I_\infty$ and $|x|<\infty$, Lipschitzian in $x$ for a function $K(t)$;*

(b) $$|x|\leqq V(t,x)\leqq K(t)|x|, \quad t\in I_\infty, \quad |x|<\infty;$$

(c) *for any solution $x(t)$ of* (1),
$$\Delta V(t,x(t))\leqq -(1-\exp(-\Delta p(t)))V(t,x(t)), \qquad t\geqq t_0.$$

This theorem will be proved by an analogous method as in differential equations, if we define a function $V(t,x)$ such that
$$V(t,x)=\sup_{\sigma\in I_\infty}|x(t+\sigma,t,x)|\, e^{p(t+\sigma)-p(t)}.$$

For the definition of the generalized exponentially asymptotic stability, see [1].

**Theorem 2.** *Suppose that*

(i) *$A(t)$ is defined for $t\in I_\infty$, and the trivial solution of* (1) *is generalized exponentially asymptotically stable;*

(ii) *$F(t,x)$ is defined for $t\in I_\infty$ and $|x|<\rho$, and $|F(t,x)|\leqq g(t,|x|)$, $t\in I_\infty$, $|x|<\rho$, where $g(t,r)$ is defined for $t\in I_\infty$ and $0\leqq r<\infty$, $g(t,0)\equiv 0$, and nondecreasing in $r$ for any $t$.*

*Then the stability or asymptotic stability of the trivial solution of*

(2) $\Delta r(t) = -(1-\exp(-\Delta p(t)))r(t) + K(t+1)g(t, r(t)),\ r(t_0) = r_0 \geqq 0$
*implies the stability or asymptotic stability of the perturbed system*

(3) $$x(t+1) = A(t)x(t) + F(t, x(t)).$$

The proof of this theorem will be completed, if we obtain an inequality $V(t, x(t, t_0, x_0)) \leqq r(t, t_0, x_0)$, provided $V(t_0, x_0) \leqq r_0$, where $V(t, x)$ is a Lyapunov function satisfying the conditions in Theorem 1.

The following result has been proved in [3] by using the properties of fundamental matrices for linear systems. But, Lyapunov function obtained in Theorem 1 can be applied to prove it.

**Theorem 3.** *Suppose that*
(i) *the trivial solution of* (1) *is exponentially asymptotically stable*;
(ii) *the function* $F(t, x)$ *satisfies an inequality* $|F(t, x)| \leqq c|x|$, $t \in I_\infty$, $|x| < \rho$ *for a sufficiently small constant* $c$.

*Then the trivial solution of* (3) *is also exponentially asymptotically stable.*

As in differential equations, the following two results show the eventual stability of the trivial solution, and Lyapunov functions will be effectively applied to prove them.

**Theorem 4.** *Suppose that*
(i) *the condition* (i) *in Theorem* 3 *is satisfied*;
(ii) $F(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, *and for any given* $\varepsilon > 0$ *there exist* $\delta(\varepsilon)$ *and* $T(\varepsilon)$ *such that*

$$|F(t, x)| \leqq \varepsilon |x|, \quad t \geqq T(\varepsilon), \quad |x| < \delta(\varepsilon);$$

(iii) $G(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, $G(t, 0) \equiv 0$, *and there exists an* $\eta > 0$ *such that*

$$|G(t, x)| \leqq \gamma(t), \quad t \in I_\infty, \quad |x| < \eta,$$

*where* $\gamma(t) \to 0$ *as* $t \to \infty$.

*Then there exists a* $T_0$ *such that, for any* $t_0 \geqq T_0$, *the trivial solution of*

$$x(t+1) = A(t)x(t) + F(t, x(t)) + G(t, x(t)), \quad x(t_0) = x_0, \quad t \geqq t_0$$

*is asymptotically stable.*

**Theorem 5.** *Suppose that*
(i) $f(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, $f(t, 0) \equiv 0$, $f_x(t, x)$ *exists, and for any given* $\varepsilon > 0$ *there exist* $\delta(\varepsilon)$ *and* $T(\varepsilon)$ *such that* $|f(t, x) - f_x(t, 0)x| \leqq \varepsilon |x|$, *whenever* $|x| < \delta(\varepsilon)$ *and* $t \geqq T(\varepsilon)$;
(ii) *the trivial solution of* $\Delta x(t) = f_x(t, 0)x(t)$ *is exponentially asymptotically stable*;
(iii) $G(t, x)$ *is defined as in* (iii) *of Theorem* 4.

*Then there exists a* $T_0$ *such that, for any* $t_0 \geqq T_0$, *the trivial solution of*

$$\Delta x(t) = f(t, x(t)) + F(t, x(t)) + G(t, x(t)),\ x(t_0) = x_0, \qquad t \geqq t_0$$

*is asymptotically stable.*

The following three results show the direct applications of Lyapunov functions to perturbed systems.

**Theorem 6.** *Suppose that*

(i) $f(t,x)$ *and* $F(t,x)$ *are defined for* $t \in I_\infty$ *and* $|x| < \infty$;

(ii) $V(t,x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \infty$, *Lipschitzian in* $x$ *for a function* $K(t)$, *and*

$$a(|x|) \leqq V(t,x) \leqq b(|x|), \quad t \in I_\infty, \quad |x| < \infty,$$

*where* $a(r)$ *and* $b(r)$ *are defined for* $0 \leqq r < \infty$, *continuous, strictly monotone increasing, and* $a(0) = b(0) = 0$;

(iii) $g(t,r)$ *is defined for* $t \in I_\infty$ *and* $0 \leqq r < \infty$, *nondecreasing in* $r$, $g(t,0) \equiv 0$, *and*

$$\Delta V(t, x(t)) \leqq g(t, V(t, x(t))), \qquad t \geqq t_0,$$

*where* $x(t)$ *is an arbitrary solution of*

$$x(t+1) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geqq t_0;$$

(iv) $w(t,r)$ *is defined for* $t \in I_\infty$ *and* $0 \leqq r < \infty$, *nondecreasing in* $r$, $w(t,0) \equiv 0$, *and*

$$|F(t,x)| \leqq w(t, |x|), \quad t \in I_\infty, \quad |x| < \infty.$$

*Then the stability properties of the trivial solution of*

$$\Delta r(t) = g(t, r(t)) + K(t+1) w(t, a^{-1}(r(t)))$$

*implies the corresponding stability properties of the trivial solution of*

$$x(t+1) = f(t, x(t)) + F(t, x(t)), \quad x(t_0) = x_0, \quad t \geqq t_0.$$

**Theorem 7.** *Suppose that*

(i) $f(t,x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, $f(t,0) \equiv 0$, $f_x(t,x)$ *exists, and for any given* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon)$ *such that* $|f(t,x) - f_x(t,0)x| \leqq \varepsilon |x|$ *uniformly in* $t \in I_\infty$, *provided* $|x| < \delta(\varepsilon)$;

(ii) $V(t,x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, *Lipschitzian in* $x$ *for a constant* $K > 0$ *and*

$$|x| \leqq V(t,x) \leqq K|x|, \quad t \in I_\infty, \quad |x| < \rho;$$

(iii) *for any solution* $y(t)$ *of* $\Delta y(t) = f_x(t,0) y(t)$, *an inequality*

$$\Delta V(t, y(t)) \leqq \alpha(t) V(t, y(t)), \qquad t \geqq t_0$$

*is satisfied, where*

$$\overline{\lim_{t\to\infty}} \frac{1}{t - t_0} \sum_{s=t_0}^{t-1} \alpha(s) < 0.$$

*Then the trivial solution of* $\Delta x(t) = f(t, x(t))$, $x(t_0) = x_0$, $t \geqq t_0$, *is asymptotically stable.*

**Theorem 8.** *Suppose that*

(i) $V(t,x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, *and*

$$|x| \leqq V(t,x) \leqq K|x|, \quad t \in I_\infty, \quad |x| < \rho,$$

*where* $K$ *is a positive constant*;

(ii) *for any solution of*

$$\Delta x(t) = f(t, x(t)) + F(t, x(t)) \tag{4}$$

*such that* $|x(t)| < \rho$, *where* $F(t,x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \rho$, *there* *holds an inequality*

$$\Delta V(t, x(t)) \leqq -c|x(t)|, \qquad t \geqq t_0,$$

*where c is a positive constant such that* $c < K$;

(iii) $w(t, r)$ *is defined for* $t \in I_\infty$ *and* $0 \leqq r < \infty$, $w(t, 0) \equiv 0$, *nondecreasing in r, and*

$$|F(t, x)| \leqq w(t, |x|), \quad t \in I_\infty, \quad |x| < \rho.$$

*Then the stability properties of the trivial solution of*

$$\Delta r(t) = -\frac{c}{K} r(t) - w(t, r(t))$$

*implies the corresponding properties of the trivial solution of* (4).

The following is a generalization of a result asserting the $l^p$-stability.

**Theorem 9.** *Suppose that there exists a function* $V(t, x)$ *satisfying the following conditions*:

(i) $V(t, x)$ *is defined and nonnegative for* $t \in I_\infty$ *and* $|x| < \infty$;

(ii) *for any solution* $x(t)$ *of* $x(t+1) = f(t, x(t))$, $x(t_0) = x_0$, *where* $f(t, x)$ *is defined for* $t \in I_\infty$ *and* $|x| < \infty$, *an inequality*

$$\Delta V(t, x(t)) + a(|x(t)|) \leqq g(t, V(t, x(t))), \qquad t \geqq t_0$$

*is satisfied, where* $a(r)$ *is the same function as before, and* $g(t, r)$ *is defined for* $t \in I_\infty$ *and* $0 \leqq r < \infty$, *and nondecreasing in r.*

*Then an inequality*

$$V(t, x(t)) + \sum_{s=t_0}^{t-1} a(|x(s)|) \leqq r(t), \qquad t \geqq t_0$$

*is fulfilled, provided* $V(t_0, x_0) \leqq r(t_0)$, *where* $r(t)$ *is a solution of* $\Delta r(t) = g(t, r(t))$, $r(t_0) = r_0$, $t \geqq t_0$.

In this result, the $l^p$-stability corresponds to the case where $a(r) = cr^p$ ($c$ is a positive constant) and $g(t, r) \equiv 0$.

## References


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