107. On the Asymptotic Behaviors of Solutions of Difference Equations. I

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1971)

Introduction. In a view point of engineering, difference equations are often used to analyze the so-called sample-data systems, in which the stability problems are considered to be very important. It seems, however, that, if we are concerned with the problems of the asymptotic behaviors for difference equations, not so many papers have been appeared so far.

The purpose of this paper is to state some results on certain types of the asymptotic behaviors of solutions of difference equations with discrete variable, but it is noted that we can obtain various results concerning the other problems, for example, the problems of boundedness, total stability, integral stability, almost periodicity, and so on, and is also expected that some results shown in this paper can be applied to the error estimation in the numerical analysis.

0. Preliminaries. We first summarize some lemmas which are often used to prove the results in the following sections. The other types of comparison theorems will be refered to [2]. We denote by I_m a set of discrete points t_0+k $(k=0, 1, \cdots, m; 0<m\leqq\infty)$, and the norms of matrices are defined suitably.

Lemma 1. *Let $g(t, r)$ be defined for $t\in I_{N-1}$ and $0\leqq r<\infty$, and nondecreasing in r for any fixed t. Then, if the function $u(t)$ and $r(t)(t\in I_N)$ satisfy the relations*

$$u(t)\leqq u_0+\sum_{s=t_0}^{t-1} g(s, u(s)), \qquad r(t)=r_0+\sum_{s=t_0}^{t-1} g(s, r(s))$$

respectively, where u_0 and r_0 are constant, there holds an inequality $u(t)\leqq r(t)$, $t\in I_N$, provided $u_0\leqq r_0$.

It is easily observed that the same result is valid, if we replace the inequality and equation by the followings respectively:

$$\Delta u(t)\leqq g(t, u(t)), \qquad \Delta r(t)=g(t, r(t)),$$

where Δ is an operator such that $\Delta\varphi(t)=\varphi(t+1)-\varphi(t)$. Such a replacement may also be applicable to Lemma 3.

Lemma 2. *Let $f(t, x)$ be defined for $t\in I_N$ and $|x|<\infty$, and $g(t, r)$ satisfy the same condition as above. Then, if an inequality $|f(t, x)|\leqq g(t, |x|)$ is satisfied, there holds an inequality $|x(t)|\leqq r(t)$, $t\in I_{N+1}$,* *provided* $|x_0|\leqq r_0$, *where* $x(t)$ *and* $r(t)$ *are the solutions*

$$\begin{matrix} \Delta x(t)=f(t,x(t)), & x(t_0)=x_0, \\ \Delta r(t)=g(t,r(t)), & r(t_0)=r_0, \end{matrix} \quad t\in I_N$$

respectively.

The following is a result corresponding to a generalized Gronwall's lemma in differential equations.

Lemma 3. *If* $u_0\geqq 0$ *is constant,* $K(t)\geqq 0$, $p(t)\geqq 0$, *and* $u(t)$ *satisfies an inequality*

$$u(t)\leqq u_0+\sum_{s=t_0}^{t-1}(K(s)u(s)+p(s)), \qquad t\in I_N,$$

the following inequalities are valid for $t\in I_N$:

$$u(t)\leqq u_0\prod_{s=t_0}^{t-1}(1+K(s))+\sum_{s=t_0}^{t-1}p(s)\prod_{\tau=s+1}^{t-1}(1+K(\tau))$$
$$\leqq u_0\exp\left(\sum_{s=t_0}^{t-1}K(s)\right)+\sum_{s=t_0}^{t-1}p(s)\exp\left(\sum_{\tau=s+1}^{t-1}K(\tau)\right).$$

1. Asymptotic behaviors. In this section, we are concerned with the asymptotic behaviors for nonlinear difference equations. In the following, for simplicity, I_∞ represents a set of nonnegative integers.

Theorem 1.1. *Let* $f(t,x)$ *be defined for* $t\in I_\infty$ *and* $|x|<\infty$, *and* $g(t,r)$ *be defined for* $t\in I_\infty$ *and* $0\leqq r<\infty$, *and nondecreasing in* r *for any fixed* t. *Suppose that for any* $t\in I_\infty$ *and* x *an inequality* $|f(t,x)|\leqq g(t,|x|)$ *is satisfied. Then, if, for any* $t_0\in I_\infty$,

(1.1) $$\Delta r(t)=g(t,r(t)), \quad r(t_0)=r_0, \quad t\geqq t_0$$

is bounded, any solution of

(1.2) $$\Delta x(t)=f(t,x(t)), \quad x(t_0)=x_0, \quad t\geqq t_0$$

such that $|x_0|\leqq r_0$ *has a limit as* $t\to\infty$.

By using Lemma 1, if we inductively obtain an inequality $|x(t)|\leqq r(t)$, $t\geqq t_0$, provided $|x_0|\leqq r_0$, the above result will be easily proved.

Theorem 1.2. *Suppose that* $g(t,r)$ *be defined as before, and any solution of* (1.1) *tends to zero as* $t\to\infty$. *Let* $f_i(t,x)(i=1,2)$ *be defined for* $t\in I_\infty$ *and* $|x|<\infty$, *and satisfy an inequality*

$$|x-y+f_1(t,x)-f_2(t,y)|\leqq|x-y|+g(t,|x-y|).$$

Then, if we denote by $x_i(t)(i=1,2)$ *the solutions of*

$$\Delta x(t)=f_i(t,x(t)), \quad x(t_0)=x_i\ (i=1,2), \quad t\geqq t_0,$$

we obtain

$$\lim_{t\to\infty}(x_1(t)-x_2(t))=0.$$

This result is a direct consequence of an inequality $|x_1(t)-x_2(t)|\leqq r(t)$, $t\geqq t_0$, provided $|x_1-x_2|\leqq r_0$.

Theorem 1.3. *Let* $g(t,r)$ *be defined for* $t\in I_\infty$ *and* $0\leqq r<\infty$, $g(t,0)\equiv 0$, *and nondecreasing in* r *for any fixed* t. *Suppose that* $f(t,x)$ *be defined for* $t\in I_\infty$ *and* $|x|<\rho$, $f(t,0)\equiv 0$, *and satisfy an inequality*

$$|x+f(t,x)|\leqq|x|+g(t,|x|), \quad t\in I_\infty, \quad |x|<\rho.$$

Then the stability properties of the trivial solution of (1.1) *implies the corresponding properties of the trivial solution of* (1.2).

This result asserts that the stability properties of the solution $r(t)$ of (1.1) implies the existence of a solution $x(t)$ of (1.2) such that $|x(t)|<\rho$, $t\geqq t_0$ and also an inequality $|x(t)|\leqq r(t)$ implies the result stated above. If $f(t,0)\not\equiv 0$, we have the following

Theorem 1.4. *Suppose that $f(t,x)$ be defined for $t\in I_\infty$ and $|x|<\rho$, and satisfy an inequality*

$$|x-y+f(t,x)-f(t,y)|\leqq|x-y|+g(t,|x-y|),$$

where $g(t,r)$ is defined as in Theorem 1.1. *Then, for a given $t_0\in I_\infty$, if any solution of*

$$\Delta r(t)=g(t,r(t))+|f(t,0)|,\quad r(t_0)=r_0\geqq 0,\quad t\geqq t_0$$

tends to zero as $t\to\infty$, every solution of (1.2) *also tends to zero as $t\to\infty$.*

The following result asserts the existence of periodic solutions, if $f(t,x)$ is periodic in t. The proof will be proceeded as in differential equations. For example, see [1].

Theorem 1.5. *Suppose that*

(i) *$g(t,r)$ is defined as in Theorem* 1.1, *and for any solution $r(t)$, $r(0)=r_0$, of* (1.1), $\lim_{t\to\infty} r(t)=0$;

(ii) *$f(t,x)$ is defined for $t\in I_\infty$ and $|x|<\infty$, periodic in t with period $\omega>0$, and satisfy an inequality*

$$|x-y+f(t,x)-f(t,y)|\leqq|x-y|+g(t,|x-y|)$$

for every t, x, and y;

(iii) *there exists a bounded solution of* (1.2).

Then the equation (1.2) *has a periodic solution of period ω.*

2. Perturbed systems. In this section, we shall consider the asymptotic behaviors of the perturbed difference equations such that

$$\begin{aligned}&(2.1)\qquad x(t+1)=A(t)x(t)+F(t,x(t)),\quad x(t_0)=x_0,\\ &(2.2)\qquad \Delta x(t)=A(t)x(t)+F(t,x(t)),\quad x(t_0)=x_0,\end{aligned}\quad t\geqq t_0,$$

and show some relations between two solutions of (2.1)(or (2.2)) and the corresponding linear equations. For the definitions of stability and the properties of fundamental matrices, see [2].

Theorem 2.1. *Let $A(t)$ be $n\times n$ matrix defined for $t\in I_\infty$,* $\det A(t)\neq 0$, *$t\in I_\infty$, and $X(t)$ be a fundamental matrix of*

$$(2.3)\qquad x(t+1)=A(t)x(t),\quad x(t_0)=x_0,\quad t\geqq t_0$$

such that $X(t_0)=E$ (the unit matrix). Let $F(t,x)$ be defined for $t\in I_\infty$ and $|x|<\infty$, $F(t,0)\equiv 0$, and satisfy an inequality $|X(t+1)^{-1}F(t,X(t)y)|\leqq g(t,|y|)$, where $g(t,r)$ is defined as in Theorem 1.1.

Suppose that any solution of (1.1) *is bounded for $t\geqq t_0$. Then, if the trivial solution of* (2.3) *is exponentially asymptotically stable, the trivial solution of* (2.3) *is also exponentially asymptotically stable.*

The following result is a direct consequence of Theorem 1.1, if we apply a transformation $x(t)=X(t)y(t)$.

Theorem 2.2. *Let* $A(t)$ *be defined as in Theorem* 2.1. *Let* $F(t,x)$ *be defined for* $t\in I_\infty$ *and* $|x|<\infty$, *and satisfy an inequality* $|X(t+1)^{-1}F(t,X(t)y)|\leqq\lambda(t)|y|$, *where* $\sum_{s=t_0}^{\infty}\lambda(s)<\infty$. *Then, for any solution* $x(t)$ *of* (2.1), *the function* $X(t)^{-1}x(t)$ *has a finite limit as* $t\to\infty$.

The concept of the asymptotic equivalency as in differential equations will be suggested by the following

Theorem 2.3. *Suppose that*

(i) $A(t)$ *is defined as in Theorem* 2.1, *and all solutions of* (2.3) *are bounded*;

(ii) $F(t,x)$ *is defined for* $t\in I_\infty$ *and* $|x|<\infty$, *and satisfies an inequality* $|F(t,x)|\leqq\lambda(t)|x|$, $t\in I_\infty$, $|x|<\infty$, *where* $\sum_{s=t_0}^{\infty}\lambda(s)<\infty$.

Then, for any solution $x(t)$ *of* (2.1), *there exists a solution* $y(t)$ *of* (2.3) *such that* $\lim_{t\to\infty}(x(t)-y(t))=0$.

If we use a quantity $\|E+A\|-1$ corresponding to the logarithmic norm in differential equations, where $\|\cdots\|$ represents a norm of matrix, we obtain the two criteria for the asymptotic stability.

Theorem 2.4. *Suppose that*

(i) $f(t,x)$ *is defined for* $t\in I_\infty$ *and* $|x|<\rho$, $f(t,0)\equiv 0$, $f_x(t,x)$ *exists, where* $f_x(t,x)$ *represents a Jacobian matrix with respect to the components of* x, *and for any given* $\varepsilon>0$, *there exists a* $\delta=\delta(\varepsilon)$ *such that* $|f(t,x)-f_x(t,0)x|\leqq\varepsilon|x|$ *uniformly in* $t\in I_\infty$, *provided* $|x|<\delta(\varepsilon)$;

(ii) *for a* $t_0\in I_\infty$,

$$\overline{\lim_{t\to\infty}}\frac{1}{t-t_0}\sum_{s=t_0}^{t-1}(\|E+f_x(s,0)\|-1)<0.$$

Then the trivial solution of $\Delta x(t)=f(t,x(t))$ *is asymptotically stable.*

Theorem 2.5. *Suppose that*

(i) $f(t,x)$ *is defined as in* (i) *of Theorem* 2.4;

(ii) *there exists a positive constant* σ *such that* $\|E+f_x(t,0)\|-1<-\sigma$, $t\in I_\infty$;

(iii) $G(t,x)$ *is defined for* $t\in I_\infty$ *and* $|x|<\rho$, $G(t,0)\equiv 0$, *and* $|G(t,x)|\leqq\gamma(t)$, $t\in I_\infty$, $|x|<\rho$, *where* $\gamma(t)\to 0$ *as* $t\to\infty$.

Then the trivial solution of $\Delta x(t)=f(t,x(t))+G(t,x(t))$ *is asymptotically stable.*

References

[1] A. Halanay: Differential Equations. Stability, Oscillations, Time Lags. Academic Press (1966).

[2] S. Sugiyama: Comparison theorems on difference equations. Bull. Sci. Engr. Research Lab., Waseda Univ., **47**, 77–82 (1970).