149. Commutative Semigroups with Greatest Group-Homomorphism

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§1. Introduction. Let S be a commutative semigroup throughout this paper. A homomorphism f of S is called a group-homomorphism of S if $f(S)$ is a group. A congruence ρ on S is called a group-congruence on S if S/ρ is a group. The smallest group-congruence ρ_0 on S is defined to be a congruence on S which is contained in all group-congruences on S. The greatest group-homomorphism $f_0: S \to G_0$ is defined to be a group-homomorphism of S onto a group G_0 having the property that for a given group-homomorphism $f: S \to G$ there is a homomorphism $h: G_0 \to G$ such that $f(x) = hf_0(x)$ for all $x \in S$. In other words $f_0: S \to G_0$ has the so-called universal repelling property with respect to homomorphisms from S onto abelian groups. The natural homomorphism $S \to S/\rho_0$ is a greatest group-homomorphism. The group G_0 is called a greatest group-homomorphic image of S, and G_0 is uniquely determined up to isomorphism. On the other hand there is a homomorphism g_1 of S into an abelian group G_1 having the universal repelling property with respect to homomorphisms from S into abelian groups. The g_1 is called a Grothendieck homomorphism (gr-homomorphism) and G_1 is called a Grothendieck group (gr-group) of S [1], [7], or a free group over S [3, Section 12.1]. For every S, g_1 always exists but f_0 does not in general. The following questions are proposed:

Under what condition on S does there exist f_0?

What structure does S have if f_0 exists?

Group-homomorphisms and group-congruences were studied by many mathematicians, Croisot [4], Dubreil [5], Levi [8], [9], Stoll [11], (also see [3]), while semigroups admitting greatest group-homomorphism have not been so much done except quite recent results by Head [6], McAlister and O'Carroll [10]. However their approach is not near to determining the structure of such commutative semigroups. The purpose of this note is to report our results to characterize commutative semigroups admitting greatest group-homomorphism in terms of (i) various kinds of homomorphisms including gr-homomorphism, (ii) conditions with respect to multiplicative structure. Theorem 3.1 gives (i) and Theorem 4.1 gives (ii). Though the complete construction of the semigroups is still open, Theorem 4.1 and 5.3 contribute to the development of this problem. Finally we give a few interesting examples. The details of this paper will be published elsewhere [13].

§ 2. Cofinal unitary subsemigroups. Group-homomorphisms of a semigroup can be characterized by unitary cofinal subsemigroups. This is a special case of the results by Dubreil [5] and others [4], [11], (see [3]). We can, however, directly obtain Lemma 2.1 below because of commutativity.

Let S be a commutative semigroup. A subsemigroup H of S is called cofinal in S if for every $x \in S$ there is an element y of S such that $xy \in H$. A subsemigroup U of S is called unitary in S if $x \in S$, $a \in U$ and $ax \in U$ implies $x \in U$. Let ρ be a group-congruence on S. The inverse image of the identity element of S/ρ under $f: S \to S/\rho$ is called the kernel of f or of ρ, denoted by Ker f or Ker ρ. It is easy to see that Ker f is unitary cofinal in S.

Lemma 2.1. *Let S be a commutative semigroup and H a cofinal subsemigroup of S. Define a relation ρ_H on S by*

$$x \rho_H y \text{ iff } ax = by \text{ for some } a, b \in H.$$

Then ρ_H is a group-congruence on S and $H \subseteq \text{Ker } \rho_H$. Furthermore $\rho_H = H$ if and only if H is unitary. There is a one-to-one correspondence between the set of all group-congruences on S and the set of all unitary cofinal subsemigroups of S by the map $\rho \to \text{Ker } \rho$.

For example all subgroups of an abelian group is unitary cofinal. Let Γ be a semilattice (i.e. commutative idempotent semigroup). Throughout this note a semilattice Γ is regarded as an upper semilattice, that is, $\alpha, \beta \in \Gamma$, $\alpha \leq \beta$ iff $\alpha\beta = \beta$. A subsemilattice Δ of Γ is cofinal in Γ if and only if for every $\alpha \in \Gamma$ there is $\beta \in \Delta$ such that $\alpha \leq \beta$.

Let S be a commutative semigroup and $S = \bigcup_{\alpha \in \Gamma} S_\alpha$ be the greatest semilattice decomposition of S. Accordingly each S_α is archimedean. A subsemigroup H of S is cofinal in S if and only if $\Delta = \{\alpha \in \Gamma : H \cap S_\alpha \neq \emptyset\}$ is cofinal in Γ.

§ 3. Characterization by homomorphisms. Let $\mathfrak{P}$ be a property described by a system of implications, for example, cancellation, separativity. A greatest $\mathfrak{P}$-homomorphism, $\mathfrak{P}$-homomorphic image and a smallest $\mathfrak{P}$-congruence are defined in a similar way as group-homomorphism and group-congruence. We note, however, that every semigroup S has a greatest $\mathfrak{P}$-homomorphism and smallest $\mathfrak{P}$-congruence. A commutative semigroup S is called separative if $a, b \in S$, $a^2 = ab = b^2$ implies $a = b$.

Theorem 3.1. *Let S be a commutative semigroup. Then the following are equivalent:*

(1) *S has a greatest group-homomorphism.*

(2) *The gr-homomorphism of S is surjective.*

(3) *The greatest cancellative homomorphic image of S is a group.*

(4) *The greatest separative homomorphic image of S has a greatest group-homomorphism.*

Let C be the greatest cancellative homomorphic image of S and Q the quotient group of (i.e. the group generated by) C. The gr-homomorphism $S \to G_1$ is the composition of the two mappings $S \to C$ and $C \to Q$. The equivalence of (2) and (3) is easily shown by this consideration. The proof of (1)$\to$(3) is obtained by Lemmas 3.2 and 3.3. The other implications are easily proved.

Lemma 3.2. *Let I be an ideal of a commutative semigroup S. Every group-homomorphism $f: I \to G$ can be uniquely extended to a group-homomorphism $h: S \to G$.*

Let $a \in \mathrm{Ker}\, f$. Define h by $xh = (ax)f, x \in S$. The following Lemma 3.3 is also obtained by Head [6], McAlister and O'Carroll [10], independently of us.

Lemma 3.3. *A commutative cancellative semigroup without idempotent does not have a greatest group-homomorphism.*

Let S be a commutative cancellative semigroup without idempotent and assume that N is the smallest unitary cofinal subsemigroup of S. Then the quotient group of N is a divisible group and N is homomorphic into the abelian group of all rational numbers. From this fact it follows that N contains smaller unitary cofinal subsemigroups of S. Every commutative semigroup D contains S as an ideal if D is not a group.

§4. Characterization by multiplicative conditions. We have

Theorem 4.1. *Let S be a commutative semigroup. The following are equivalent*

(1) *S has a greatest group-homomorphism.*

(2) *For every $x, y \in S$ there exist $z, u \in S$ such that $xzu = yu$.*

(3) *For every $a \in S$ there exist $b, c \in S$ such that $abc = c$.*

The equivalence (1)$\rightleftarrows$(2) is immediate by Theorem 3.1. The proof of (3)$\to$(2) is also easy, but we need some preparation to prove one of (1) and (2) implies (3). First we assume S is separative. Let $S = \bigcup_{\alpha \in \Gamma} S_\alpha$ be the greatest semilattice decomposition of S. Each S_α is archimedean and cancellative (see [2]). Then S can be embedded into a semilattice of groups, $Q = \bigcup_{\alpha \in \Gamma} Q_\alpha$, where Q_α is the quotient group of S_α (see [2]). We can see that Q has a greatest group-homomorphism f_0 and that if S has a greatest group-homomorphism, then $S \cap \mathrm{Ker}\, f_0 \neq \emptyset$ and it is cofinal in S. (3) is just a restatement of this result. In other words (3) holds if and only if $\{x \in S : xy = y$ for some $y \in S\}$ is not empty and cofinal in S. Thus (3) can be derived under the assumption that S is separative. However we can easily exclude the condition of separativity.

§5. **Cofinal subsemigroups and group-congruences.** Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be defined as in the proof of Theorem 4.1. If Λ is a subsemilattice of Γ, then $T = \bigcup_{\alpha\in\Lambda} S_\alpha$ is called a *cluster* in S.

Proposition 5.1. *Let W be a cofinal cluster in a commutative semigroup S. A group-homomorphism f of W onto G can be uniquely extended to a homomorphism $\bar{f}$ of S onto G such that $f = \bar{f}\,|\,W$.*

Proposition 5.2. *Let W be a cofinal cluster in S. If W has a greatest group-homomorphism, then also does S.*

Theorem 5.3. *A commutative semigroup S which is not archimedean has a greatest group-homomorphism if and only if there is a proper cofinal cluster W in S such that W has a greatest group-homomorphism.*

In particular if Γ is finite, S has a greatest group-homomorphism if and only if the ideal archimedean component of S contains an idempotent. This is also obtained by [10], and the case where S is archimedean is done by [6] independently of us.

§6. **Examples.** The following two are simplest examples of commutative semigroups admitting greatest group-homomorphism.

(1) A semilattice of abelian groups.

(2) A commutative semigroup in which the set of all idempotents form a cofinal subband.

We are, however, more interested in commutative separative semigroups without idempotent.

Let G be an abelian group and Γ the set of all positive integers and, for $i, j \in \Gamma$ let $i+j$ denote the usual addition, but let $i\cdot j = \text{Max.}\{i, j\}$.

Let $S = \{(x, \alpha, i) : x, i \in \Gamma, \alpha \in G\}$. We define in S three operations as follows:

Example 1.

$$(x,\alpha,i)(y,\beta,j) = \begin{cases} (x+y, \alpha\beta, i) & i=j \\ (x, \alpha\beta, i) & i>j \\ (y, \alpha\beta, j) & i<j \end{cases}$$

Example 2.

$$(x,\alpha,i)(y,\beta,j) = \begin{cases} (x+y, \alpha\beta, i) & i=j \\ (x, \alpha, i) & i>j \\ (y, \beta, j) & i<j \end{cases}$$

Example 3.

$$(x,\alpha,i)(y,\beta,j) = (x+y, \alpha\beta, i\cdot j)$$

In any case above S is a chain of S_α, $S = \bigcup_{\alpha\in\Gamma} S_\alpha$, where Γ is a chain $\{1, 2, 3, \cdots\}$ and each S_α is a copy of the direct product of the additive semigroup $\{1, 2, 3, \cdots\cdot\}$ and the group G. Examples 1 and 2 have a greatest group-homomorphic image, G and the trivial group, respectively, but Example 3 does not have.

Addendum. After writing this paper the author found that the proof of Theorem 4.1 is much simplified if (3) of Theorem 3.1 is used, that is, (3) of Theorem 4.1 is an immediate consequence of (3) of Theorem 3.1.

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