172. A Remark on Multiplicative Linear Functionals on Measure Algebras

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1. Let G be a non-discrete locally compact abelian group with the dual group Γ. Let $\bar{\Gamma}^B$ be the Bohr compactification of Γ. Let $M(G)$ be the Banach algebra consisting of all bounded regular Borel measures on G under the convolution multiplication and $\mathfrak{M}$ the maximal ideal space of $M(G)$. By $\hat{\mu}$ we denote the Gelfand transform of $\mu \in M(G)$. We may suppose that Γ is the open subset of $\mathfrak{M}$. Let $\bar{\Gamma}$ be the closure of Γ in $\mathfrak{M}$.

In [1], E. Hewitt and S. Kakutani showed the following theorem.

Theorem 0 (E. Hewitt and S. Kakutani). *If H is a compact subgroup of G and $A[H]$ is a subalgebra of $M(G)$ consisting of all measures which are absolutely continuous with respect to the Haar measure on H, there is a multiplicative linear functional f in $\bar{\Gamma}\backslash\Gamma$ such that $\hat{\mu}(f)=\mu(G)$ for all $\mu \in A[H]$.*

Let $\mathfrak{J}$ be a σ-ring generated by cosets of H and $M(\mathfrak{J})$ a subalgebra of $M(G)$ of all measures which are concentrated on $\mathfrak{J}$, to prove Theorem 0, it is enough to show that there is a multiplicative linear functional f such that $\hat{\mu}(f)=\mu(G)$ for all $\mu \in M(\mathfrak{J})$. It is reasonable to conjecture that this theorem is true under more weak hypothesis, that is, H is a non-open closed subgroup of G. Since $M(\mathfrak{J})^{\perp}$, which is the subspace consisting of all measures that are singular with respect to all measures in $M(\mathfrak{J})$, is an ideal, there is a multiplicative linear functional f_0 such that $\hat{\mu}(f_0)=\mu(G)$ if $\mu \in M(\mathfrak{J})$ and $\hat{\mu}(f_0)=0$ if $\mu \in M(\mathfrak{J})^{\perp}$. Then, it is natural to conjecture that f_0 is an element of $\bar{\Gamma}\backslash\Gamma$.

In this paper, we shall show that these conjectures are true.

2. We may suppose that $\bar{\Gamma}^B$ is the compact subset of $\mathfrak{M}$ as follows:

$$\hat{\mu}(\gamma)=\int_G(-x,\gamma)d\lambda(x) \qquad (\gamma \in \bar{\Gamma}^B,\ \mu \in M(G))$$

where λ is the discrete part of μ. Throughout this section, for $\mu \in M(G)$ let λ be the discrete part of μ and η the continuous part of μ.

At first, we shall show the following theorem.

Theorem 1. *$\bar{\Gamma}^B$ is contained in $\bar{\Gamma}\backslash\Gamma$.*

Proof. Suppose that $\{V_\alpha\}$ is a neighborhood base of 0 in G, for each V_α there is a continuous positive definite function f_α whose compact support lies in V_α such that $f_\alpha(0)=1$, and define

$$A_\alpha(\mu)=\int_\Gamma \hat{f}_\alpha(\gamma)\,|\hat{\mu}(\gamma)|^2\,d\gamma \qquad (\mu\in M(G)).$$

We say that $\lim_\alpha A_\alpha(\mu)=A$ if to every $\varepsilon>0$ there is a neighborhood V of 0 in G such that $|A_\alpha(\mu)-A|<\varepsilon$ for all $V_\alpha\subset V$. We can have

$$\lim_\alpha A_\alpha(\mu)=\sum_{x\in G}|\mu(\{x\})|^2 \qquad \text{for any } \mu\in M(G) \text{ ([2]).}$$

Define the canonical continuous injection φ of Γ into $\bar{\Gamma}^B$ such that $\hat{\mu}(\varphi(\gamma))=\hat{\lambda}(\gamma)$ for $\mu\in M(G)$ and $\gamma\in\Gamma$. Then $\varphi(\Gamma)$ is the dense subgroup of $\bar{\Gamma}^B$. Since $\bar{\Gamma}\backslash\Gamma$ is closed, it is enough for our purpose to prove that $\varphi(\Gamma)\subset\bar{\Gamma}\backslash\Gamma$. Given $\gamma_0\in\Gamma$, $\varepsilon>0$ and $\mu_1,\cdots,\mu_m\in M(G)$. Put

$$V=\bigcap_{k=1}^m\{f\in\mathfrak{M}:|\hat{\mu}_k(f)-\hat{\mu}_k(\varphi(\gamma_0))|<\varepsilon\}.$$

If we can prove that $V\cap\Gamma\neq\phi$, this completes the proof. Let

$$V'=\bigcap_{k=1}^m\{\gamma\in\Gamma:|\hat{\eta}_k(\gamma)|+|\hat{\lambda}_k(\gamma)-\hat{\lambda}_k(\gamma_0)|<\varepsilon\}.$$

Clearly, $V\cap\Gamma\supset V'$. We shall prove $V'\neq\phi$. Assume $V'=\phi$. Put

$$U=\bigcap_{k=1}^m\{\gamma\in\bar{\Gamma}^B:|\hat{\mu}_k(\gamma)-\hat{\mu}_k(\varphi(\gamma_0))|<\varepsilon/2\}.$$

Obviously,

$$U=\bigcap_{k=1}^m\{\gamma\in\bar{\Gamma}^B:|\hat{\lambda}_k(\gamma)-\hat{\lambda}_k(\gamma_0)|<\varepsilon/2\}.$$

Then, since U is open in $\bar{\Gamma}^B$ and $\varphi(\Gamma)$ is the dense subgroup of $\bar{\Gamma}^B$, there is a finite subset $\{\gamma_1,\cdots,\gamma_n\}$ of Γ such that

$$(1)\qquad \bigcup_{i=1}^n((\varphi(\gamma_i))+U)=\bar{\Gamma}^B.$$

Put $W=\varphi^{-1}(U)$, then

$$W=\bigcap_{k=1}^m\{\gamma\in\Gamma:|\hat{\lambda}_k(\gamma)-\hat{\lambda}_k(\gamma_0)|<\varepsilon/2\}.$$

Furthermore, by (1)

$$(2)\qquad \Gamma=\bigcup_{i=1}^n(\gamma_i+W).$$

We put

$$W_k=\{\gamma\in\Gamma:|\hat{\eta}_k(\gamma)|>\varepsilon/2\} \qquad (k=1,2,\cdots,m),$$

then since $V'=\phi$, we have that $\bigcup_{k=1}^m W_k\supset W$. Thus by (2) it follows that

$$(3)\qquad \bigcup_{i=1}^n\bigcup_{k=1}^m(\gamma_i+W_k)=\Gamma.$$

On the other hand, from $\int_\Gamma\hat{f}_\alpha(\gamma)d\gamma=1, \hat{f}_\alpha\geqq 0$ and (3) we can get that

$$(4)\qquad \int_{(\gamma_{i(\alpha)}+W_{k(\alpha)})}\hat{f}_\alpha(\gamma)d\gamma\geqq\frac{1}{mn}$$

for some choice $i(\alpha)\in\{1,\cdots,n\}$ and $k(\alpha)\in\{1,\cdots,m\}$. Put

$$\eta_{i,k}(E)=\int_G(x,\gamma_i)\chi_E d\eta_k(x)$$

for every Borel subset E of G. Clearly, $\eta_{i,k}$ is a continuous measure for each i and k, thus

(5) $$\lim_{\alpha} \sum_{i=1}^{n} \sum_{k=1}^{m} A_\alpha(\eta_{i,k})=0.$$

However, from (4) it follows that

$$\begin{aligned}\sum_{i=1}^{n} \sum_{k=1}^{m} A_\alpha(\eta_{i,k}) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_\Gamma \hat{f}_\alpha(\gamma)\,|\hat{\eta}_{i,k}(\gamma)|^2\, d\gamma \\ &\geqq \int_\Gamma \hat{f}_\alpha(\gamma)\,|\hat{\eta}_{i(\alpha),k(\alpha)}(\gamma)|^2\, d\gamma \\ &\geqq \int_{(\gamma_{i(\alpha)}+W_{k(\alpha)})} \hat{f}_\alpha(\gamma)^{\varepsilon^2}/4d\gamma \geqq \varepsilon^2/4mn \geqq 0.\end{aligned}$$

This is contradict to (5). Thus, $V' \neq \phi$. This completes the proof.

Theorem 2. *Let H be a non-open closed subgroup of G. Let $\mathfrak{J}$ be the σ-ring generated by all cosets of H. We denote by $M(\mathfrak{J})$ the closed subalgebra of $M(G)$ consisting of measures that are concentrated on $\mathfrak{J}$. We denote by $M(\mathfrak{J})^\perp$ the complementary ideal of $M(\mathfrak{J})$. Define the linear functional f_0 on $M(G)$ as follows:*

$$\hat{\mu}(f_0)=\begin{cases}\mu(G) & \text{if } \mu \in M(\mathfrak{J}),\\ 0 & \text{if } \mu \in M(\mathfrak{J})^\perp.\end{cases}$$

Then f_0 is an element of $\bar{\Gamma}\backslash\Gamma$.

Proof. It is evident that f_0 is a multiplicative linear functional on $M(G)$. Let Λ is the anihilator of H in Γ. As well known, Λ is the dual group of G/H. Since G/H is non-discrete, Λ is non-compact. Let ψ be the canonical homomorphism of G to G/H, then there is a homomorphism Φ of $M(G)$ *onto* $M(G/H)$ such that $\Phi\mu(E)=\mu(\psi^{-1}(E))$ for each Borel set E of G/H ([2]). If $\mathfrak{M}_H$ is the maximal ideal space of $M(G/H)$, then Φ induces the continuous injection α of $\mathfrak{M}_H$ into $\mathfrak{M}$ such that

(6) $$\hat{\mu}(\alpha f)=\widehat{\Phi\mu}(f) \qquad (f \in \mathfrak{M}_H,\ \mu \in M(G)).$$

Clearly, $\alpha(\Lambda)\subset\Gamma$. Easily, we can get that

(7) $$\Phi(M(\mathfrak{J}))=M_d(G/H) \quad \text{and} \quad \Phi(M(\mathfrak{J})^\perp)=M_c(G/H),$$

where $M_d(G/H)$ and $M_c(G/H)$ are the subalgebra consisting of all discrete and continuous measures on G/H respectively. Define a multiplicative linear functional g_0 on $M(G/H)$ such that

$$\hat{\mu}(g_0)=\begin{cases}\mu(G/H) & \text{if } \mu \in M_d(G/H),\\ 0 & \text{if } \mu \in M_c(G/H).\end{cases}$$

Then, from Theorem 1, let $\bar{\Lambda}$ be the closure of Λ in $\mathfrak{M}_H$, g_0 is contained in $\bar{\Lambda}/\Lambda$. Thus, (6) and (7) show $\alpha g_0=f_0$. Hence, from that α is continuous and that Λ is closed in Γ, we have that $f_0 \in \bar{\Gamma}\backslash\Gamma$. This completes the proof.

Let H be a non-open closed subgroup of G. Then there is the weakest locally compact topology τ on G such that H is an open subgroup of a locally compact abelian group (G,τ). Let Γ_H be the dual group of (G,τ). For $\gamma \in \Gamma_H$, define a multiplicative linear functional on $M(G)$ as follows:

$$\hat{\mu}(\gamma)=\begin{cases}\int_G(-x,\gamma)d\mu(x) & \text{if } \mu\in M(\mathfrak{J}),\\ 0 & \text{if } \mu\in M(\mathfrak{J})^{\perp}.\end{cases}$$

Then, Γ_H may be considered a subset of $\mathfrak{M}$. Furthermore, if $\gamma\in\Gamma_H$ and $\mu\in M(\mathfrak{J})$, then $\gamma d\mu\in M(\mathfrak{J})$. Thus, it follows the next corollary to Theorem 2.

Corollary. *Γ_H is contained in $\bar{\Gamma}\backslash\Gamma$.*

References

[1] E. Hewitt and S. Kakutani: Some multiplicative linear functionals on $M(G)$. Annals of Math., **79**, 489–505 (1964).

[2] W. Rudin: Fourier Analysis on Groups. Interscience, New York (1962).