5. *Results Related to Closed Images of M-Spaces. III*

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1. Introduction. Throughout this paper by a space we shall mean a T_1-space, and by N the set of natural numbers. For a space X let us consider the following conditions, where the same terminology as in [9] will be used.

(CM): There exists a sequence $\{\mathfrak{F}_n \mid n \in N\}$ of hereditarily closure-preserving closed covers of X such that

(i) any sequence $\{A_n\}$ with $x \in A_n \in \mathfrak{F}_n$ for $n \in N$ is either hereditarily closure-preserving or a q-sequence at a point x of X, and

(ii) every point x of X has a q-sequence $\{A_n\}$ with $x \in A_n \in \mathfrak{F}_n$ for $n \in N$.

(qk) X is a quasi-k-space (Nagata [11]).

(q) X is a q-space in the sense that each point of X has a q-sequence which consists of neighborhoods of x (Michael [5]).

(sst) X is semi-stratifiable (cf. Creede [2]).

(σ) X is a σ-space in the sense that there is a σ-locally finite network for X (Okuyama [13]).

As is known, (σ) implies (sst) and (q) implies (qk) if X is regular (cf. [6, Theorem 2. F. 2]), but (q) does not imply (qk) if X is Hausdorff (cf. [6, Example 10. 11]).

The purpose of this paper is to prove the following theorems except Theorem 1.1 which was obtained in [9] and is stated here for comparison.

Theorem 1.1. *A regular space X is the closed image of a regular M-space iff* (CM) *and* (qk) *hold.*

Theorem 1.2. *A Hausdorff space X is the closed image of a metric space iff* (CM), (qk) *and* (sst) *(or* (σ)*) hold.*

Theorem 1.3. *A Hausdorff space X is metrizable iff* (CM), (q) *and* (sst) *(or* (σ)*) hold.*

Theorem 1.4. *A space X is an M^*-space iff* (CM) *and* (q) *hold.*

Theorem 1.5. *A regular space X is semi-metrizable iff* (q) *and* (sst) *hold.*

In view of Theorem 1.4, Theorem 1.3 implies Theorem 1.6 below, which is due to Ishii and Shiraki [4] for (sst) and to Shiraki [15] for (σ),[1] but we shall first give a new proof of the latter and then make

1) I have heard from J. Nagata that F. Slaughter proved that a Hausdorff space is metrizable iff it is an M-space and a σ-space.

use of it for the proof of the former.

Theorem 1.6. *A Hausdorff space X is metrizable iff X is an M^*-space and* (sst) (*or* (σ)) *holds.*

Following Michael [6], we shall call a space X singly bi-quasi-k (resp. countably bi-quasi-k) if for any subset F of X (resp. any decreasing sequence $\{F_n \mid n \in N\}$ of subsets of X) with $x \in \mathrm{Cl}\, F$ (resp. $x \in \mathrm{Cl}\, F_n$ for $n \in N$) there is a decreasing q-sequence $\{A_n\}$ at x with $x \in \mathrm{Cl}\,(F \cap A_n)$ (resp. $x \in \mathrm{Cl}\,(F_n \cap A_n)$) for $n \in N$ such that every sequence $\{x'_n\}$ with $x'_n \in A_n$ has a cluster point in $\cap A_n$. Then we have

Theorem 1.1* ([9]). *A regular space X is the closed image of a regular M-space iff X is singly bi-quasi-k and* (CM) *holds.*

Theorem 1.4*. *A space X is an M^*-space iff X is countably bi-quasi-k and* (CM) *holds.*

2. Basic lemmas.

Lemma 2.1. *Let X be a countably paracompact space and $f: X \to Y$ a closed continuous onto map. Then* $\mathrm{Bd}\, f^{-1}(y)$ *is countably compact for $y \in Y$ if Y is countably bi-quasi-k or a q-space.*[2]

Proof. Let $y \in Y$. Suppose that there is a discrete closed set $\{x_n \mid n \in N\}$ with $x_n \in \mathrm{Bd}\, f^{-1}(y)$. Let us put $G_n = X - \{x_j \mid j \neq n\}$ and $G_0 = X - \{x_n \mid n \in N\}$. Then $\{G_i \mid i = 0, 1, \cdots\}$ is a countable open cover of X and hence has a locally finite open refinement $\{U_i \mid i = 0, 1, \cdots\}$ such that $U_i \subset G_i$ for each i. Then

$$x_n \in U_n,\ y \in \mathrm{Cl}\,(f(U_n) - y) \quad \text{for} \quad n \in N.$$

Put $F_n = \cup \{f(U_i) - y \mid i \geqq n\}$. Then there is a decreasing q-sequence $\{A_n\}$ at y such that $y \in \mathrm{Cl}\,(F_n \cap A_n)$ for $n \in N$. Hence there are distinct points $y_{k(n)}$ of Y with $k(n) \in N$, such that

$$y_{k(n)} \in (f(U_{k(n)}) - y) \cap A_{k(n-1)+1} \quad \text{and} \quad k(n) < k(n+1) \quad \text{for} \quad n \in N,$$

where we put $k(0) = 0$.

Then $\{y_{k(n)}\}$ has a cluster point but this is a contradiction since $\{U_{k(n)}\}$ is locally finite. This proves Lemma 2.1.

Lemma 2.2. *Let X be a space satisfying* (CM). *Then there exist a metric space B, an M-space S which is a closed subset of $B \times X$, and a continuous onto map $f: S \to X$ such that*

(i) *f is a closed map or a closed map with* $\mathrm{Bd}\, f^{-1}(x)$ *countably compact for each point x of X according as*

(a) (qk) *holds, or*

(b) *X is countably bi-quasi-k or a q-space;*

(ii) *S is paracompact in case every countably compact closed subset of X is compact.*

Proof. Define, exactly as in the proof of [9, Theorem 3.1 and Proposition 5.2], a metric space B, an M-space $S \subset B \times X$ and a map

2) Cf. Michael [6, Theorem 9.1].

$f: S \to X$; we shall use the same notation as there except that Y and X there are replaced here by X and S.

Take a closed subset A of X and let $x_0 \in \mathrm{Cl}\, f(A) - f(A)$. Then we can find indices $\alpha_n \in \Omega_n$ for $n \in N$ such that for every $n \in N$,

$$x_0 \in \mathrm{Cl}\, [f(A \cap (B(\alpha_1, \cdots \alpha_n) \times X)] \subset \bigcap_{i=1}^{n} F_{i\alpha_i}.$$

Assume that X is countably bi-quasi-k or a q-space. Then there is a decreasing q-sequence $\{A_n\}$ at x_0 such that

$$x_0 \in \mathrm{Cl}\, [f(A \cap (B(\alpha_1, \cdots, \alpha_n) \times X)) \cap A_n] \quad \text{for} \quad n \in N.$$

Since X is T_1, there are distinct points $x_n \in X, n \in N$, such that

$$x_n \in f(A \cap (B(\alpha_1, \cdots, \alpha_n) \times X)) \cap A_n \subset F_{n\alpha_n}.$$

Then $\{x_n\}$ has a cluster point and hence $\{F_{n\alpha_n} | n \in N\}$ must be a q-sequence at x_0. Hereafter, similarly as before, we can conclude that f is a closed map. In this case by Lemma 2.1 $\mathrm{Bd}\, f^{-1}(x)$ is countably compact for each point x of X since an M-space is countably paracompact (cf. [3]). The other assertions were proved in [9], and so this completes the proof.

3. Proof of Theorems 1.4 and 1.4*. The "only if" part follows from the definition of M^*-spaces and the "if" part is a direct consequence of Lemma 2.2.

4. Proof of Theorem 1.6. The "only if" part is obvious. Suppose that X is an M^*-space and that (sst) or (σ) hold. Since every countably compact space satisfying (sst) or (σ) is compact (cf. Creede [2]), in the present case the space S in Lemma 2.2 is paracompact and Hausdorff. Since S is semi-stratifiable (or a σ-space), so is $S \times S$. Hence S is metrizable by Okuyama [12] and Borges [1]. Hence by Stone [14] and Morita-Hanai [8] X is metrizable.

5. Proof of Theorem 1.2. Since the closed image of a semi-stratifiable space is semi-stratifiable by Creede [2, Theorem 3.1] (for the case of σ-spaces, cf. Okuyama [13]), the "only if" part is a direct consequence of Theorem 1.1. To prove the "if" part, suppose that X satisfies conditions (CM), (qk) and (sst) (or (σ)). Then the space S in Lemma 2.2 satisfies (sst) or (σ) in the present case. Therefore by Theorem 1.6 S is metrizable. This completes the proof in view of Lemma 2.2.

6. Proof of Theorem 1.5. Suppose that X satisfies (q) and (sst). Let x be a point of X. Then there is a q-sequence $\{U_n | n \in N\}$ of open neighborhoods of x such that $\cap \{U_n | n \in N\} = x$ and $\mathrm{Cl}\, U_{n+1} \subset U_n$ for $n \in N$. Clearly $\{U_n\}$ is a basis for neighborhoods at x. Hence X is first-countable. Therefore by Creede [2, Corollary 1.4] X is semi-metrizable. The "only if" part follows also from the same result of [2].

7. Proof of Theorem 1.3. Theorem 1.3 is now a direct consequence of Theorems 1.4 and 1.6.

8. Remarks. (1) Theorems 1.3 and 1.6 and the "if" part of Theorem 1.2 remain true if we replace (sst) (or (σ)) by any topological property (P) such that (a) every metric space has (P), and (P) is preserved under taking closed subsets and products with metric spaces; (b) a countably compact Hausdorff space with (P) is compact; (c) a paracompact Hausdorff M-space with (P) has a G_δ-diagonal. This is obvious from §§ 4, 5 and 7. As an example of such a property (P) we can mention the property of a space having a point-countable pseudobase (=separating open cover) (cf. Shiraki [15] and Michael-Slaughter [16]).

(2) A space X satisfying (CM) is a P-space in the sense of [10]. Because, if $\{G(\alpha_1, \cdots, \alpha_i) \mid \alpha_j \in \Omega, j=1, \cdots, i; i \in N\}$ is a family of open sets of X such that $G(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$, then for the family $\{F(\alpha_1, \cdots, \alpha_i)\}$ of F_σ-sets defined by

$$F(\alpha_1, \cdots, \alpha_i) = \cup \{F \in \cup \mathfrak{F}_n \mid F \subset G(\alpha_1, \cdots, \alpha_i)\},$$

$X = \bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)$ implies $X = \bigcup_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)$ (indeed, if $\{A_n\}$ is a q-sequence at x in condition (CM) and if $\cap A_n \subset G(\alpha_1, \cdots, \alpha_i)$ then we have $A_n \subset G(\alpha_1, \cdots, \alpha_i)$ for some $n \in N$).

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