1. *On Polarizations of Certain Homogeneous Spaces*

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1. It is one of main problems in the theory of unitary representations to find a unified way of constructing all irreducible unitary representations for an arbitrary Lie group.

Kostant ([7], [8]) has shown a very general method of constructing unitary representations, using polarizations of homogeneous sympletic spaces. And Kirillov ([5]) gave several problems related with Kostant's works.

In this note, we announce some results on invariant polarizations of homogeneous spaces by semi-simple Lie groups. First, an infinitesimal characterization of polarizations is given in Theorem 1. And examples in the third section would answer to one of the problems raised by Kirillov. Then the notion of principal nilpotent elements in a real semi-simple Lie algebra is introduced. Using this notion, we can show the existence of invariant polarizations of certain homogeneous spaces (Theorems 3 and 4). An exposition with detailed proofs will be published elsewhere.

2. Let G be a connected Lie group, $\mathfrak{g}_0$ its Lie algebra and $\mathfrak{g}_0'$ the dual vector space of $\mathfrak{g}_0$. The space $\mathfrak{g}_0'$ has the G-module structure contragredient to the adjoint representation of G on $\mathfrak{g}_0$. For an element f of $\mathfrak{g}_0$, we denote by G^f the isotropy subgroup of G with respect to f, and by $\mathfrak{g}_0^f$ the subalgebra of $\mathfrak{g}_0$ corresponding to G^f. Kostant has shown that every G-orbit $G(f)=G/G^f$ has a canonical invariant symplectic structure.

Let $\mathfrak{g}$ and $\mathfrak{g}^f$ be the complexification of $\mathfrak{g}_0$ and $\mathfrak{g}_0^f$. For a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we consider the following conditions:

i) $\mathfrak{g}^f \subset \mathfrak{p}$;

ii) $f([\mathfrak{p}, \mathfrak{p}])=\{0\}$;

iii) $\dim \mathfrak{p} - \dim \mathfrak{g}^f = \dim \mathfrak{g} - \dim \mathfrak{p}$;

iv) $\mathfrak{p}$ is $Ad(G^f)$-stable;

v) $\mathfrak{p}+\sigma(\mathfrak{p})$ is a complex subalgebra of $\mathfrak{g}$, where σ denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Definition. For f in $\mathfrak{g}_0'$ and a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, $\mathfrak{p}$ is called

1) *a weak polarization* of f if $\mathfrak{p}$ satisfies i)—iii),

2) *a polarization* of f if $\mathfrak{p}$ satisfies i)—iv),

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3) *a weak admissible polarization* of f if $\mathfrak{p}$ satisfies i)—iii) and v) and

4) *an admissible polarization* of f if $\mathfrak{p}$ satisfies i)—v).

It is easily seen that our (admissible) polarization in the above definition corresponds to an invariant (admissible) polarization of the homogeneous space G/G^f with respect to the canonical invariant symplectic structure given by Kostant. Whereas, a weak polarization of f corresponds to an invariant polarization of the universal covering space of G/G^f, considered as a homogeneous space by the universal covering group of G.

Throughout this paper we assume that G is a connected semi-simple Lie group. In this case, the G-module $\mathfrak{g}_0'$ is isomorphic to $\mathfrak{g}_0$ via the Killing form. For an element X of $\mathfrak{g}_0$ and a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we call $\mathfrak{p}$ a polarization (resp. a weak polarization etc.) of X if $\mathfrak{p}$ is a polarization (resp. a weak polarization etc.) of f_X, where f_X is the element of $\mathfrak{g}_0'$ corresponding to X by the above isomorphism.

3. The following theorem gives a characterization of a weak polarization.

Theorem 1. *For X in $\mathfrak{g}_0$ and a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, the following conditions are equivalent:*

1) *$\mathfrak{p}$ is a weak polarization of X;*

2) *$\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and the space $[X, \mathfrak{p}]$ coincides with the nilradical of $\mathfrak{p}$.*

Particularly, in case that X is a nilpotent element of $\mathfrak{g}_0$, the above conditions are equivalent to

3) *X belongs to the orthogonal complement of $\mathfrak{p}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$, and $\mathfrak{p}$ satisfies the condition* iii) *in the second section.*

An element X in $\mathfrak{g}_0$ has a unique decomposition $X = H + e$ such that H is semi-simple, e is nilpotent and $[H, e] = 0$. We remark here that the centralizer $\mathfrak{g}^H$ of H is reductive and e belongs to the semi-simple part $[\mathfrak{g}^H, \mathfrak{g}^H]$ of $\mathfrak{g}^H$.

Theorem 2. *Let $X = H + e$ be the decomposition of X as above. Then X has a weak polarization if and only if e has a weak polarization in the semi-simple part of the centralizer of H in $\mathfrak{g}$.*

Proposition 1. *Any semi-simple element of $\mathfrak{g}_0$ has an admissible polarization.*

Thus the problem to find a polarization for an arbitrary element X in a real semi-simple Lie algebra is reduced to the case where X is nilpotent. However, the following proposition and remark suggest that the situation is somewhat complicated.

Proposition 2. *Suppose that $\mathfrak{g}_0$ is a direct sum of simple Lie algebras of type (A). Then every element in $\mathfrak{g}_0$ has a weak polarization.*

Remark. In case that a certain simple factor of $\mathfrak{g}_0$ is the normal form of type (B), (C), (D), (E), (F) or (G), there exists an element in $\mathfrak{g}_0$ with no weak polarizations.

In the following two examples, we fix a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the highest root α relative to a certain linear order in Δ, where Δ denotes the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Example 1. $\mathfrak{g}_0 = o(3,2)$.
Since $\mathfrak{g}_0$ is the normal form of type (B_2), there exists a non-zero element X in $\mathfrak{g}_0$ such that $ad(H)X = \alpha(H)X$ for every H in $\mathfrak{h}$. Using Theorem 1, one can easily see that this element X has not a weak polarization, since $\dim \mathfrak{g}^X = 6$, $\dim \mathfrak{g} = 10$, and the dimension of a parabolic subalgebra of $\mathfrak{g}$ is equal to 6, 7 or 10.

Example 2. The normal form of type (G_2).
There exists a non-zero element X in $\mathfrak{g}_0$ such that $ad(H)X = \alpha(H)X$ for every H in $\mathfrak{h}$. This element X has not a weak polarization, since $\dim \mathfrak{g}^X = 8$, $\dim \mathfrak{g} = 14$, and the dimension of a parabolic subalgebra of $\mathfrak{g}$ is equal to 8, 9 or 14.

4. Let θ be a Cartan involution of $\mathfrak{g}_0$, and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ associated to θ, where $\mathfrak{k}_0$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h}_0^1, \cdots, \mathfrak{h}_0^k$ be representatives of the G-conjugate classes of Cartan subalgebras of $\mathfrak{g}_0$. They can be chosen θ-stable and such that $\mathfrak{h}_-^i \subset \mathfrak{h}_-^1, \mathfrak{h}_+^i \subset \mathfrak{h}_+^k$ and $\dim \mathfrak{h}_+^i \leqq \dim \mathfrak{h}_+^{i+1}$ for every i, where $\mathfrak{h}_-^i = \mathfrak{h}_0^i \cap \mathfrak{k}_0$ and $\mathfrak{h}_+^i = \mathfrak{h}_0^i \cap \mathfrak{p}_0$. We set $\mathfrak{h}^i = (\mathfrak{h}_0^i)^c$ and $\mathfrak{h}_R^i = \sqrt{-1}\,\mathfrak{h}_-^i + \mathfrak{h}_+^i$. The non-zero root system Δ^i of $(\mathfrak{g}, \mathfrak{h}^i)$ admits the direct sum decomposition $\Delta^i = \Sigma_k^i \cup \Sigma_p^i \cup \Lambda$, where $\Sigma_{\mathfrak{k}}^i = \{\alpha \in \Delta^i\,;\, \mathfrak{g}^\alpha \subset k\}$, $\Sigma_{\mathfrak{p}}^i = \{\alpha \in \Delta^i\,;\, \mathfrak{g}^\alpha \subset \mathfrak{p}\}$ and $\Lambda^i = \{\alpha \in \Delta^i\,;\, \alpha \,|\, \mathfrak{h}_+^i \neq 0\}$. A lexicographic order in $\mathfrak{h}_R^i$ compatible to $\mathfrak{h}_+^i$ induces a linear order in Δ^i and determines positive subsystems Δ_+^i and Λ_+^i. We put $\mathfrak{n}_0^i = (\sum_{\alpha \in \Lambda_+^i} \mathfrak{g}^\alpha) \cap \mathfrak{g}_0$. Then we have

Proposition 3. *A nilpotent element in $\mathfrak{g}_0$ is G-conjugate to an element in $\mathfrak{n}_0^k$.*

Proposition 4. 1) *An element $X \in \mathfrak{g}_0$ is G-conjugate to an element in $\mathfrak{h}_0^i + \mathfrak{n}_0^i$ for some i,*

2) *In the above, assume that i is chosen such that* $\dim \mathfrak{h}_+^i$ *is as large as possible. And let X' be an element in $\mathfrak{h}_0^i + \mathfrak{n}_0^i$ which is G-conjugate to X, and H' be the semisimple part of X'. Then H' satisfies $\alpha(H') \neq 0$ for every $\alpha \in \Sigma_{\mathfrak{p}}^i$.*

5. Now we shall give the notion of principal nilpotent elements (in the real sense) of $\mathfrak{g}_0$:

Definition. A nilpotent element $e \in \mathfrak{g}_0$ is called *a principal nilpotent element* if $\dim \mathfrak{g}^e \leqq \dim \mathfrak{g}^X$ for every nilpotent element X in $\mathfrak{g}_0$, where $\mathfrak{g}^X$ denotes the centralizer of X in $\mathfrak{g}$.

The dimension of $\mathfrak{g}^e$ for a principal nilpotent element $e \in \mathfrak{g}$ are written explicitly:

Proposition 5. *A nilpotent element* $e \in \mathfrak{g}_0$ *is principal nilpotent if and only if* $\dim \mathfrak{g}^e$ *is equal to the rank of* $\mathfrak{g}_0$ *plus the cardinal number of* $\Sigma_{\mathfrak{k}}^k$.

Following two theorems are concerned with cuspidal parabolic subalgebras and polarizations:

Theorem 3. *A principal nilpotent element of* $\mathfrak{g}_0$ *has an admissible polarization* $\mathfrak{q}$, *which can be chosen so that* $\mathfrak{q} \cap \mathfrak{g}_0$ *may be a minimal parabolic subalgebra of* $\mathfrak{g}_0$.

Theorem 4. *Assume that* $H \in \mathfrak{h}_0^i$ *satisfies* $\alpha(H) \neq 0$ *for every* $\alpha \in \Sigma_{\mathfrak{p}}^i$. *Then there exists an element* $e \in \mathfrak{n}_0^i$ *such that*

$$\mathfrak{q} = \mathfrak{h}^i + \sum_{\alpha \in \Delta_+^i} \mathfrak{g}^\alpha + \sum_{\substack{\alpha \in \Sigma_{\mathfrak{k}}^i \\ \alpha > 0 \\ \alpha(H) = 0}} \mathfrak{g}^{-\alpha}$$

is a weak admissible polarization of $H + e$.

Remark. For the parabolic subalgebra $\mathfrak{q}$ in Theorem 4, $(\mathfrak{q} + \sigma\mathfrak{q}) \cap \mathfrak{g}_0$ is a cuspidal parabolic subalgebra of $\mathfrak{g}_0$ corresponding to $\mathfrak{h}_0^i$.

References

[1] L. Auslander and B. Kostant: Quantization and representations of solvable Lie groups. Bull. Amer. Math. Soc., **73**, 692–695 (1967).

[2] R. Bott: Homogeneous vector bundles. Ann. of Math., **66**, 203–248 (1957).

[3] Harish-Chandra: Harmonic analysis on semisimple Lie groups. Bull. Amer. Math. Soc., **76**, 529–551 (1970).

[4] Kirillov: Unitary representations of nilpotent Lie groups. Russian Math. Survey., **17**, 53–104 (1962).

[5] Kirillov, A. A.: Construction of irreducible unitary representations of Lie groups. Vestnik Mosc. Univ., **2**, 41–51 (1970).

[6] B. Kostant: The principal three dimensional subgroup and the Betti numbers of a complex simple Lie group. Amer. J. Math., **81**, 973–1032 (1959).

[7] ——: Orbits, Symplectic Structures and Representation Theory. Proc. U.S.-Japan Sem. on Diff. Geom. Kyoto, 71 (1965).

[8] ——: Quantization and Unitary Representations. Lectures in Modern Analysis and Applications. III. Lecture Notes, **170**, 87–208 (1970). Springer-Verlag.

[9] J. A. Wolf: The action of a real semisimple group on a complex flag manifold. I; Orbit structure and holomorphic arc components. Bull. Amer. Math. Soc., **75**, 1121–1237 (1969).