29. The Theory of Nuclear Spaces Treated by the Method of Ranked Space. V

By Yasujirô NAGAKURA
Science University of Tokyo

(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1972)

§6. The dual space. In the previous papers [3]-[5] and [6], we investigated the extended nuclear space $\hat{\Phi}$, that is, the space to which the countably Hilbertian nuclear space was enlarged by the method of ranked space given by K. Kunugi.

In this paper we shall study the dual space of $\hat{\Phi}$.

In the sequel, without loss of generality, we may take up only a fundamental sequence of neighbourhoods of a simple type $\{\hat{V}_{\gamma(i)}(0)\}$ with $\gamma(i)\leqq\gamma(i+1)$ and $\gamma(i)\to\infty$, where $\hat{V}_{\gamma(i)}(0)$ denotes $\hat{V}_{\gamma(i)}(0, 1/\gamma(i), \gamma(i))$.

Definition 9. We say that a linear functional F defined on the space $\hat{\Phi}$ is R-continuous if $\hat{g}_n \xrightarrow{R} \hat{g}$ in $\hat{\Phi}$ implies $\lim_{n\to\infty} F(\hat{g}_n)=F(\hat{g})$. Furthermore, let $\hat{\Phi}'$ be the set of all R-continuous linear functionals on $\hat{\Phi}$ and let it be called the dual space of $\hat{\Phi}$.

From now on we shall write $P_i(g)$ in place of $P_{i,i}(g)$.

Since the set $M_i=\{g\in\Phi;\ P_i(g)=0\}$ is a subspace in Φ, we have $\Phi=N_i\oplus M_i$, where

$$N_i=\left\{g\in\Phi;\ \left\|\sum_{k=i+1}^{\infty}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}=0\right\}.$$

This means that every element g in Φ is represented by $g=g^{(1)}+g^{(2)}$ such that

$$g^{(1)}=\sum_{k=1}^{i}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}$$

and

$$g^{(2)}=\sum_{k=i+1}^{\infty}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}.$$

Lemma 28. *We have* $N_1\subsetneqq N_2\subsetneqq\cdots\subsetneqq N_i\subsetneqq\cdots$.

Proof. In general, we shall prove $N_i\subsetneqq N_{i+1}$, where

$$N_i=\left\{\sum_{k=1}^{i}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}};\ g\in\Phi\right\}$$

and

$$N_{i+1}=\left\{\sum_{k=1}^{i+1}\lambda_{k,n_i,n_{i+1}}(g,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i};\ g\in\Phi\right\}.$$

Since we have

$$\varphi_{h,n_i}=\sum_{k=1}^{\infty}\lambda_{k,n_{i-1},n_i}(\varphi_{h,n_i},\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}},$$

and then we obtain $\varphi_{h,n_i}=\lambda_{h,n_{i-1},n_i}\varphi_{h,n_{i-1}}$.

Lemma 29.

(1) *To every* $g\in\Phi$, *we have* $\varepsilon_i P_i(g)\leqq\varepsilon_{i+1}P_{i+1}(g)$.

(2) *To every* $g\in N_i$, *there exists a constant number*

$$a(i)>1 \quad \textit{such that} \quad P_{i+1}(g)\leqq a(i)P_i(g).$$

Proof. (1) To every $g\in\Phi$, we have

$$g=\sum_{k=1}^{\infty}\lambda_{k,n_i,n_{i+1}}(g,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i}.$$

Now we obtain

$$\begin{aligned}
\varepsilon_i P_i(g)&=\left\|\sum_{k=1}^{i}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}\\
&\leqq\left\|\sum_{k=1}^{i+1}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}\\
&=\left\|\sum_{k=1}^{i+1}\lambda_{k,n_{i-1},n_i}\left(\sum_{h=1}^{\infty}\lambda_{h,n_i,n_{i+1}}(g,\varphi_{h,n_{i+1}})_{n_i+1}\varphi_{h,n_i},\varphi_{k,n_i}\right)_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}\\
&=\left\|\sum_{k=1}^{i+1}\lambda_{k,n_{i-1},n_i}\lambda_{k,n_i,n_{i+1}}(g,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}\\
&=\left\|\sum_{k=1}^{i+1}\lambda_{k,n_i,n_{i+1}}(g,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i}\right\|_{n_{i-1}}\\
&\leqq\left\|\sum_{k=1}^{i+1}\lambda_{k,n_i,n_{i+1}}(g,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i}\right\|_{n_i}=\varepsilon_{i+1}P_{i+1}(g)
\end{aligned}$$

(2) If an element g belongs to N_i, that is, $(g,\varphi_{k,n_i})_{n_i}=0$ for all $k>i$, we have

$$\begin{aligned}
P_i^2(g)&=\left\|\sum_{k=1}^{i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}^2\\
&\geqq\left\{\min_{k=1\cdots i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2\right\}\sum_{k=1}^{i}|(g,\varphi_{k,n_i})_{n_i}|^2.
\end{aligned}$$

Since Φ_{n_i} is a Hilbert space with the orthonormal system $\{\varphi_{k,n_i}\}_k$, any element $g'\in\Phi_{n_i}$ is represented by $g'=\sum_{k=1}^{\infty}(g',\varphi_{k,n_i})_{n_i}\varphi_{k,n_i}$. On the other hand, we have

$$g'=\sum_{k=1}^{\infty}\lambda_{k,n_i,n_{i+1}}(g',\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i}.$$

And then we obtain

$$\lambda_{k,n_i,n_{i+1}}(g',\varphi_{k,n_{i+1}})_{n_{i+1}}=(g',\varphi_{k,n_i})_{n_i}.$$

So that, we have

$$\sum_{k=1}^{i}|(g,\varphi_{k,n_i})_{n_i}|^2=\sum_{k=1}^{i}|\lambda_{k,n_i,n_{i+1}}(g,\varphi_{k,n_{i+1}})_{n_{i+1}}|^2=\varepsilon_{i+1}^2P_{i+1}^2(g).$$

Consequently we obtain

$$P_i^2(g)\geqq\left\{\min_{k=1\cdots i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2\right\}\varepsilon_{i+1}^2P_{i+1}^2(g).$$

Now, by § 3 in [4], we know

$$2\left(\sum_{k=1}^{\infty}\lambda_{k,n_{i-1},n_i}\right)\varepsilon_{i+1}\leqq\varepsilon_i,$$

and then we have

$$1<2\left(\sum_{k=1}^{\infty}\lambda_{k,n_{i-1},n_i}\right)\Big/\min_{k=1\cdots i}(\lambda_{k,n_{i-1},n_i})<\varepsilon_i\Big/\varepsilon_{i+1}\left\{\min_{k=1\cdots i}(\lambda_{k,n_{i-1},n_i})\right\}.$$

If we set $a(i)=\varepsilon_i/\varepsilon_{i+1}\{\min_{k=1\cdots i}(\lambda_{k,n_{i-1},n_i})\}$, we obtain (2). Q.E.D.

Lemma 30. *To every $g\in\hat{\Phi}$, we have*

$$\varepsilon_i\hat{P}_i(g)\leqq\varepsilon_{i+1}\hat{P}_{i+1}(g).$$

Proof. It is clear.

Lemma 31. *The restriction of the semi-norm P_i to N_i is a norm on N_i And the restriction of the semi-norm $P_j(j>i)$ to N_i is equivalent to the norm P_i on N_i.*

Proof. By Lemma 21 in [6], we have

$$P_i(g)=\left\|\sum_{k=1}^{i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}$$

for every $g\in\Phi$.

If an element $g^{(1)}$ belongs to N_i, there exists some element $g\in\Phi$ such that

$$g^{(1)}=\sum_{k=1}^{i}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}.$$

So that, we have

$$P_i(g^{(1)})=\left\|\sum_{k=1}^{i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)\left(\sum_{h=1}^{i}\lambda_{h,n_{i-1},n_i}(g,\varphi_{h,n_i})_{n_i}\varphi_{h,n_{i-1}},\varphi_{k,n_i}\right)_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}.$$

On the other hand, the proof of Lemma 28 shows $\varphi_{k,n_i}=\lambda_{k,n_{i-1},n_i}\varphi_{k,n_{i-1}}$. Hence we obtain

$$P_i(g^{(1)})=\left\|\sum_{k=1}^{i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}=\|g^{(1)}\|_{n_{i-1}}/\varepsilon_i.$$

The other half of the lemma is evident by Lemma 29. Thus the proof is complete.

In the papers [5] and [6], we showed that the linear ranked space $\hat{\Phi}$ is the completion of the nuclear space Φ by the method of ranked space. Now, we identify Φ with $\hat{\Phi}_0$ and regard Φ as a dense subset in $\hat{\Phi}$.

Lemma 32. *The set $\bigcup_{i=1}^{\infty}N_i$ is dense in $\hat{\Phi}$, that is, for every element $g\in\hat{\Phi}$, there exists a sequence $\{g_i\}$ with $g_i\in N_i$ such that $g_i\xrightarrow{R}g$.*

For every element $g\in\hat{\Phi}$, there exists a sequence $\{f_i\}$ in Φ such that $f_i\xrightarrow{R}g$. This means that there exists some fundamental sequence of neighbourhoods $\{\hat{V}_{\gamma(i)}(0)\}$, $\gamma(i)\leqq\gamma(i+1)$, $\gamma(i)\to\infty$ such that $f_i-g\in\hat{V}_{\gamma(i)}(0)$ for all integer i. On the other hand, we have $f_i=g_i+g_i'$ with $g_i\in N_{\gamma(i)}$ and $g_i'\in M_{\gamma(i)}$ for every integer i. And then we have

$$\begin{aligned}\hat{P}_{\gamma(i)}(g_i-g)&=\hat{P}_{\gamma(i)}(g_i+g_i'-g-g_i')\leqq\hat{P}_{\gamma(i)}(f_i-g)+\hat{P}_{\gamma(i)}(g_i')\\&=\hat{P}_{\gamma(i)}(f_i-g)+P_{\gamma(i)}(g_i')=\hat{P}_{\gamma(i)}(f_i-g)\leqq1/\gamma(i),\end{aligned}$$

hence we obtain $g_i-g\in\hat{V}_{\gamma(i)}(0,2/\gamma(i),\gamma(i))$. Consequently we see

$$g_i-g\in\hat{V}_{\gamma'(i)}(0),\quad\text{where}\quad\gamma'(i)=\left[\frac{\gamma(i)}{2}\right]$$

for integer i such that $[\gamma(i)/2]>1$. This means $g_i\xrightarrow{R}g$.

Lemma 33. *Every R-continuous linear functional F defined on*

$\hat{\Phi}$ is uniquely determined on $\bigcup_{i=1}^{\infty} N_i$.

Proof. It is clear.

Lemma 34. *The restriction F_i to N_i of F, which is an R-continuous linear functional defined on $\hat{\Phi}$, is continuous with respect to the norm P_i on N_i.*

Proof. Let a sequence $\{g_n\}$ belong to the set N_i and coverge to an element g in N_i with respect to the norm P_i on N_i. And then by Lemma 29, the sequence $\{g_n\}$ converges to g with respect to all $P_i(i=1,2,\cdots)$. Hence for every neighbourhood $V_i(0,1/i,i)\equiv V_i(0)$, there exists some number N such that the relation $n\geqq N$ implies $g_n-g \in V_i(0)$. Consequently we have $g_n \xrightarrow{R} g$. Hence we obtain $F_i(g_n) \to F_i(g)$. This proof is complete.

Definition 10. We shall define the inner product $(g, \varphi_{k,n_i})_{n_i}$ for a given element g in $\hat{\Phi}$. There exists some R-cauchy sequence of elements $\{g_\xi\}$ in Φ such that $g_\xi \xrightarrow{R} g$. And then to any $\hat{V}_i(0,r,m)$, there exists some integer N such that the relations $\xi\geqq N$ and $\eta\geqq N$ imply $g_\xi-g_\eta \in \hat{V}_i(0,r,m)$.

Since g_ξ and g_η belong to Φ, we have $g_\xi-g_\eta \in V_i(0,r,m)$, that is

$$P_{i,m}^2(g_\xi-g_\eta)=\Big\|\sum_{k=1}^{m}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g_\xi-g_\eta,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\Big\|_{n_{i-1}}^2$$
$$=\sum_{k=1}^{m}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2|(g_\xi,\varphi_{k,n_i})_{n_i}-(g_\eta,\varphi_{k,n_i})_{n_i}|^2<r^2.$$

Hence $\{(g_\xi,\varphi_{k,n_i})_{n_i}\}_\xi$ is a Cauchy sequence of numbers.

Then we define $(g,\varphi_{k,n_i})_{n_i}=\lim_{\xi\to\infty}(g_\xi,\varphi_{k,n_i})_{n_i}$.

Consequently we have

$$\hat{P}_{i,m}(g)=\lim_{\xi\to\infty}P_{i,m}(g_\xi)=\lim_{\xi\to\infty}\Big\|\sum_{k=1}^{m}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g_\xi,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\Big\|_{n_{i-1}}$$
$$=\Big\|\sum_{k=1}^{m}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\Big\|_{n_{i-1}}.$$

Theorem 5. *Let F be an R-continuous linear functional defined on $\hat{\Phi}$, then there exists some integer i_0 such that the relation $i_0<i$ implies $F(g)=0$ for $g\in M_i$, where $M_i=\{\varphi\in\Phi;\ P_i(\varphi)=0\}$.*

Proof. Suppose that this theorem is not true. From

$$P_i(\varphi)=\Big\|\sum_{k=1}^{i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(\varphi,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\Big\|_{n_{i-1}},$$

there exists some sequence of integers $\{\gamma(i)\}$ such that $\gamma(i)<k_{\gamma(i)}<\gamma(i+1)$ and $F(\varphi_{k_{\gamma(i)},n_{\gamma(i)-1}})=\alpha_i\neq 0$.

Then we have $(\varphi_{k_{\gamma(i)},n_{\gamma(i)-1}}/\alpha_i)\xrightarrow{R}0$, because $(\varphi_{k_{\gamma(i)},n_{\gamma(i)-1}}/\alpha_i)\in V_{\gamma(i)}(0)$ for every integer i.

This is a contradiction for $F(\varphi_{k_{\gamma(i)},n_{\gamma(i)-1}}/\alpha_i)=1$.

Lemma 35. *Let M_i^0 be the annihilator in $\hat{\Phi}'$ of M_i. Then,*

(1) *M_i^0 is a finite dimensional subspace in $\hat{\Phi}'$:*

(2) $M_i^0 \subseteq M_{i+1}^0$

Proof. (1) Since we have

$$M_i = \left\{ \sum_{k=i+1}^{\infty} \lambda_{k,n_{i-1},n_i}(g, \varphi_{k,n_i})_{n_i} \varphi_{k,n_{i-1}} ; g \in \Phi \right\},$$

we obtain

$$M_i^0 = \left\{ F \in \hat{\Phi}' ; F\left(\sum_{k=i+1}^{\infty} \lambda_{k,n_{i-1},n_i}(g, \varphi_{k,n_i})_{n_i} \varphi_{k,n_{i-1}} \right) = 0, g \in \Phi \right\}.$$

Hence it follows $M_i^0 = \{F \in \hat{\Phi}' : F(\varphi_{k,n_{i-1}}) = 0 \text{ for } k > i\}$.

Consequently the element of M_i^0 corresponds to $\{\alpha_1, \cdots, \alpha_i\}$ such that $F(\varphi_{k,n_{i-1}}) = \alpha_k$ for $k = 1, \cdots, i$.

(2) From $M_i \supseteq M_{i+1}$, it is evident.

Theorem 6. *We have* $\hat{\Phi}' = \bigcup_{i=1}^{\infty} M_i^0$.

Proof. By Theorem 5, it is clear.

Lemma 36. *Let F be an R-continuous linear functional in $\hat{\Phi}$. Then, there exists some integer i such that $F \in M_i^0$ and*

$$F(g) = \sum_{k=1}^{i} \lambda_{k,n_{i-1},n_i}(g, \varphi_{k,n_i})_{n_i} F(\varphi_{k,n_{i-1}})$$

for all $g \in \hat{\Phi}$.

Proof. By Theorem 5, there exists some integer i such that $F \in M_i^0$. And to every element $g \in \hat{\Phi}$ there exists some sequence of elements $\{g_\xi\}$ in Φ such that $g_\xi \xrightarrow{R} g$. Thus we have $F(g_\xi) \to F(g)$.

On the other hand, since we have

$$g_\xi = \sum_{k=1}^{\infty} \lambda_{k,n_{i-1},n_i}(g_\xi, \varphi_{k,n_i})_{n_i} \varphi_{k,n_{i-1}},$$

we obtain

$$F(g_\xi) = \sum_{k=1}^{i} \lambda_{k,n_{i-1},n_i}(g_\xi, \varphi_{k,n_i})_{n_i} F(\varphi_{k,n_{i-1}}).$$

Consequently we see

$$F(g) = \lim_{\xi \to \infty} F(g_\xi) = \lim_{\xi \to \infty} \left\{ \sum_{k=1}^{i} \lambda_{k,n_{i-1},n_i}(g_\xi, \varphi_{k,n_i})_{n_i} F(\varphi_{k,n_{i-1}}) \right\}$$
$$= \sum_{k=1}^{i} \lambda_{k,n_{i-1},n_i}(g, \varphi_{k,n_i})_{n_i} F(\varphi_{k,n_{i-1}}).$$

Definition 11. We define

$$V^*(0, h, i) = \{F \in M_i^0 : \sup_{g \in \hat{V}_i(0,1,i)} |F(g)| < \varepsilon_i / h\},$$

where h is a positive integer, as a neighbourhood of the origin in $\hat{\Phi}'$ and we call it a neighbourhood of rank h. Furthermore we define that the neighbourhood with rank 0, V_0^* is always the space $\hat{\Phi}'$.

Lemma 37. *We have* $V^*(0, h, i) \subseteq V^*(0, h, j)$ *if* $i \leqq j$.

Proof. Let F belong to $V^*(0, h, i)$. Then we have $|F(g)| < \varepsilon_i / h$ for every $g \in \hat{V}_i(0, 1, i)$, i.e. $\hat{P}_i(g) < 1$. Now, if f belongs to $\hat{V}_j(0, 1, j)$, i.e., $\hat{P}_j(f) < 1$, we see $\hat{P}_i(\varepsilon_i f / \varepsilon_j) < 1$ by Lemma 30. Thus we obtain $|F(\varepsilon_{if} / \varepsilon_j)| < \varepsilon_i / h$. This shows that F belongs to $V^*(0, h, j)$.

Lemma 38. *We have* $i \leqq j$ *if* $V^*(0, l, i) \subseteq V^*(0, h, j)$ *with* $l \geqq h$.

Proof. Suppose $i>j$. Put F to be a linear functional such that

(1) $F(\varphi_{k,n_{i-1}})=0$ for $k=1\cdots j\cdots,i-1,i+1,\cdots$

(2) $F(\varphi_{i,n_{i-1}})=1/2l$.

Then F does not belong to $V^*(0,h,j)$, but belongs $V^*(0,l,i)$. Because to every $g\in\hat{V}_i(0,1,i)$, we have

$$|F(g)|=\left|\sum_{k=1}^{i}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})_{n_i}F(\varphi_{k,n_{i-1}})\right|$$
$$\leqq\varepsilon_i\left(\sum_{k=1}^{i}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2|(g,\varphi_{k,n_i})_{n_i}|^2\right)^{1/2}|F(\varphi_{i,n_{i-1}})|$$
$$<\varepsilon_i|F(\varphi_{i,n_{i-1}})|<\varepsilon_i/l.$$

References

[1] I. M. Gel'fand and N. Ya Vilenkin: Generalized Functions, Vol. 4 (1964).

[2] K. Kunugi: Sur la méthode des espaces rangés. I, II. Proc. Japan Acad., **42**, 318–322, 549–554 (1966).

[3] Y. Nagakura: The theory of nuclear spaces treated by the method of ranked space. I. Proc. Japan Acad., **47**, 337–341 (1971). (1971).

[4] ——: The theory of nuclear spaces treated by the method of ranked space. II. Proc. Japan Acad., **47**, 342–345 (1971).

[5] ——: The theory of nuclear spaces treated by the method of ranked space. III. Proc. Japan Acad., **47** (Supplement) (1971).

[6] ——: The theory of nuclear spaces treated by the method of ranked space. IV. Proc. Japan Acad., **47** (Supplement) (1971).