19. On Quasi-Fibrations over Spheres

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1. Let X be a CW-complex of the form $S^k \bigcup_\alpha e^n \bigcup_\beta e^{n+k}$. X is called a quasi-fibration over S^n if there exists a map

$$p:(X, S^k) \to (S^n, pt)$$

which induces homotopy isomorphisms. On the other hand we have the notion of k-spherical fibrations over S^n in the sense of Hurewicz.

Let $q: E \to S^n$ be a k-spherical fibration so that it is known that the pair (E, S^k) has the homotopy type such as (X, S^k). It is clear that, if (X, S^k) has the homotopy type of a pair (E, S^k) X is a quasi-fibration over S^n. In this note we shall prove the following

Theorem 1.1. *For a CW-complex X of the form $S^k \bigcup_\alpha e^n \bigcup_\beta e^{n+k}$ $(n \geqq k+2 \geqq 4)$. Let $p:(X, S^k) \to (S^n, pt)$ be a quasi-fibration. Then the pair (X, S^k) has the homotopy type of a pair of a k-spherical fibration over S^n.*

Remark. Probably, the condition $n \geqq k+2$ can be removed. Let $\tilde{\alpha} \in \pi_n(S^k \bigcup_\alpha e^n, S^k)$ be the generator which $\partial(\tilde{\alpha}) = \alpha$, let $\iota_k \in \pi_k(S^k)$ be the generator and let $i: S^k \to S^k \bigcup_\alpha e^n$ and $j: S^k \bigcup_\alpha e^n \to (S^k \bigcup_\alpha e^n, S^k)$ be the inclusions respectively.

For the proof of theorem we need following lemmas.

Lemma 1.2. *The pair (X, S^k) $(n \geqq k+2 \geqq 4)$ has the homotopy type of a pair of a k-spherical fibration over S^n if and only if*

$$j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r,$$

where $[,]_r$ denotes the relative Whitehead product.

Lemma 1.3. *Let $p:(X, S^k) \to (S^n, pt)$ be a quasi-fibration $(n \geqq k+2 \geqq 4)$. Then we have $j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r$.*

It is obvious that Theorem 1.1 follows from lemmas.

Moreover, Theorem 2.1 in [1] shows that the existence of a quasi-fibration follows from the condition $j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r$. Hence we have

Collorary 1.4. *For $X = S^k \bigcup_\alpha e^n \bigcup_\beta e^{n+k}$ $(n \geqq k+2 \geqq 4)$, X has the homotopy type of the total space of a k-spherical fibration over S^n if and only if $j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r$, or there exists a quasi-fibration $p:(X, S^k) \to (S^n, pt)$.*

2. In this section we shall give the proofs of lemmas. First we prove Lemma 1.3. Let $Q: S^k \bigcup_\alpha e^n \to S^n$ be the natural collapsing map. By a theorem of Blaker-Massy we know that

$$\pi_{n-1+k}(S^k \bigcup_\alpha e^n, S^k) = Z[\tilde{\alpha}, \iota_k]_r + \pi_{n-1+k}(S^n), \tag{2.1}$$

where the second component is determined by Q_*.

Suppose that there exists a quasi-fibration

$$p : (X, S^k) \to (S^n, pt).$$

Then it is clear that Q is extendable over X so that it holds

$$j_*(\beta) = m[\tilde{\alpha}, \iota_k]_r \tag{2.2}$$

for some integer m.

Consider the homotopy sequence of the tripple $(X, S^k \bigcup_\alpha e^n, S^k)$

$$\pi_{n+k}(X, S^k) \longrightarrow \pi_{n+k}(X, S^k \bigcup_\alpha e^n) \xrightarrow{\partial} \pi_{n+k-1}(S^k \bigcup_\alpha e^n, S^k).$$

$\pi_{n+k}(X, S^k \bigcup_\alpha e^n)$ is the infinite cyclic group generated by $\tilde{\beta}$ such as $\partial\tilde{\beta} = j_*(\beta)$ and, by $\pi_{n+k}(X, S^k) = \pi_{n+k}(S^n)$, $\pi_{n+k}(X, S^k)$ is finite so that ∂ is an isomorphism. Consider the composite of homomorphisms

$$\pi_{n+k-1}(S^k \bigcup_\alpha e^n, S^k) \xrightarrow[i_*]{} \pi_{n+k-1}(X, S^k) \xrightarrow[p_*]{} \pi_{n+k-1}(S^n).$$

Since $p_* i_*([\tilde{\alpha}, \iota_k]_r) = 0$ and p_* is isomorphic so we have $i_*[\tilde{\alpha}, \iota_k]_r = 0$. Hence there exists an element $\gamma \in \pi_{n+k}(X, S^k \bigcup_\alpha e^n)$ such that

$$\partial\gamma = [\tilde{\alpha}, \iota_k]_r.$$

Let $\gamma = s\beta$. Since $\gamma = s\beta = sj_*(\beta) = sm[\tilde{\alpha}, \iota_k]_r$ by (2.2) we have

$$[\tilde{\alpha}, \iota_k]_r = ms[\tilde{\alpha}, \iota_k]_r.$$

Then from (2.1), we obtain $m = +1$, i.e. the proof is completed. Secondarily we consider Lemma 1.2. Suppose that (X, S^k) has the homotopy type of (E, S^k), where $q : E \to S^n$ is a k-spherical fibration. As explained in § 1, there exists a CW-complex X_E of the form $S^k \bigcup_a e^n \bigcup_b e^{n+k}$ and (E, S^k) has the homotopy type of (X_E, S^k). By (4.1) of 2, we have

$$j_*(\tilde{b}) = +[\tilde{a}, \iota_k]_r. \tag{2.3}$$

Let $w : (X, S^k) \to (X_E, S^k)$ be the homotopy equivalence by assumption. It is trivial that $w_*(\iota_k) = \pm\iota_k$, $w_*(\tilde{\alpha}) = \pm\tilde{a}$ and $w_*(\beta) = \pm\tilde{b}$. Hence, by (2.3), we have

$$\begin{aligned} w_* j_*(\beta) = j_* w_*(\beta) &= \pm j_*(\tilde{b}) = \pm[\tilde{a}, \iota_k]_r \\ &= \pm[w_*(\tilde{\alpha}), w_*(\iota_k)]_r = \pm w_*[\tilde{\alpha}, \iota_k]_r. \end{aligned}$$

Since w_* is isomorphic we obtain $j_*(\beta) = \pm[\alpha, \iota_k]_r$.

Next we assume that $j_*(\beta) = \pm[\tilde{\alpha}, \iota_k]_r$ for X. Let G_k be the space consisted of homotopy equivalences of S^k with degree 1 and let $p_k : G_k \to S^k$ be the canonical fibration with the fibre F_k. Since $0 = \Delta j_*(\beta) = \pm[\alpha, \iota_k]$ there exists an element $\chi \in \pi_{n-1}(G_k)$ such that $p_k(\chi) = \alpha$ by Lemma 1.1 in [2]. Let $q_\chi : E_\chi \to S^n$ be the k-spherical fibering with the characteristic class χ. Then $K_\chi = S^k \bigcup_\alpha e^n \bigcup_\gamma e^{n+k}$. Consider the homotopy sequence of the pair $(S^k \bigcup_\alpha e^n, S^k)$

$$\pi_{n-1+k}(S^k) \xrightarrow[i_*]{} \pi_{n-1+k}(S^k \bigcup_\alpha e^n) \xrightarrow[j_*]{} \pi_{n-1+k}(S^k \bigcup_\alpha e^n, S^k).$$

By suitable orientations we have

$$j_*(\beta) - j_*(\gamma) = [\tilde{\alpha}, \iota_k]_r - [\tilde{\alpha}, \iota_k]_r = 0.$$

Hence there exists an element $\sigma \in \pi_{n-1+k}(S^k)$ such that

$$i_*(\sigma)+\gamma=\beta \tag{2.4}$$

Let $\tau \in \pi_{n-1}(G_k)$, $\xi \in \pi_{n-1}(F_k)$ and K_τ, K_ξ be the complexes which are obtained from the k-spherical fibering with the characteristic class τ, ξ respectively. Let α_τ, γ_τ be the attaching class for the complex K such as α, γ for K_χ and let $\tau'=\tau+i_{k*}(\xi)$, where $i_k: F_k \to G_k$ is the inclusion. Since $p_{k*}(\tau')=p_{k*}(\tau)$, γ_τ and $\gamma_{\tau'}$ are elements of $\pi_{n-1+k}(S^k \bigcup_\mu e^n)$, where $\mu=p_{k*}(\tau)$. Then we have

Lemma 2.1. *Let $\lambda: \pi_{n-1}(F_k) \to \pi_{n-1+k}(S^k)$ be the isomorphism defined by B. Steer* [4] *then it holds $\gamma_{\tau'}=\gamma_\tau+\lambda(\xi)$.*

For let $h: S^n \to S_1^n \vee S_2^n$ be a map with type (1.1) and let

$$P: E \to S_1^n \vee S_2^n$$

be a k-spherical fibering whose restrictions satisfy $P|S_1^n=p_{i_{k*}}(\xi)$, and $P|S_2^n=p_\tau$. The fibration induced from E by h is clearly the fibration with the characteristic class τ'. Let $H: K_{\tau'} \to E_E$ be the map induced by h. Obviously we have

$$H_*(\gamma_{\tau'})=i_*^1(\gamma_\tau)+i_*^2(\gamma_{i_{k*}(\xi)}) \tag{2.5}$$

where i^1 and i^2 denote inclusions: $e_1^n \cup S^k, S^k \cup e_2^n \to e_1^n \bigcup_{\alpha_{\xi'}} S^k \bigcup_{\alpha_\tau} e_2^n$ and $\xi'=i_{k*}(\xi)$. Since K_E has a decomposition

$$e_1^{n+k} \bigcup_{\gamma_{\xi'}} e_1^n \bigcup_{\alpha_{\xi'}} S^k \bigcup_{\alpha_\tau} e_2^n \bigcup_{\gamma_\tau} e_2^{n+k}$$

and $\alpha_{\xi'}=p_{k*}(\xi')=p_{k*}(i_{k*}(\xi))=0$, $S^k \bigcup_{\alpha_{\tau'}} e^n=S^k \bigcup_{\alpha_\tau} e^n$ is a retract of $e_1^n \bigcup_{\alpha_{\xi'}} S^k \bigcup_{\alpha_\tau} e_2^n$. By applying the retraction to (2.5) we obtain

$$\gamma_{\tau'}=\gamma_\tau+\lambda(\xi)$$

from Lemma 3.2 in [2] (it states that $\gamma_{\xi'}=\lambda(\xi)+[\iota_k, \iota_n]$).

3. Addendum. In § 1 we noted that the condition $n \geqq k+2$ was probably removed. In fact, if $\alpha=0$ ($X=S^k \vee S^n \bigcup_\beta e^{n+k}$) and there exists a quasi-fibration $p: (X, S^k) \to (S^n, pt)$, X has the homotopy type of a total space of a k-spherical fibering over S^n. This is so if $2 \leqq n \leqq k$. The proof is as follows; Since

$$\pi_{n-1+k}(S^k \vee S^n)=\pi_{n-1+k}(S^k)+\pi_{n-1+k}(S^n)+\boldsymbol{Z}[\iota_k, \iota_n]$$

we have $\beta=\beta_1+\beta_2+m[\iota_k, \iota_n]$ for $\beta_1 \in \pi_{n-1+k}(S^k)$, $\beta_2 \in \pi_{n-1+k}(S^n)$ and some integer m.

The existence of p shows that the second component β_2 vanishes. Now, from the commutative diagram (exact)

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
\boldsymbol{Z} \approx \pi_{n+k}(X, S^k \vee S^n) & \to & \pi_{n-1+k}(S^k \vee S^n, S^k) & \to & \pi_{n-1+k}(X, S^k) \\
 & \searrow & \uparrow & & \wr\wr \\
 & & \pi_{n-1+k}(S^k \vee S^n) & & \pi_{n-1+k}(S^n) \\
 & & \uparrow & & \\
 & & \pi_{n-1+k}(S^k) & & \\
 & & \uparrow & & \\
 & & 0 & &
\end{array}$$

we can obtain $m=\pm 1$. By choosing a suitable orientation we can suppose that $\beta=\beta_1+[\iota_k, \iota_n]$. Let $q: E \rightarrow S^n$ be the k-spherical fibering with the characteristic class $\lambda^{-1}(\beta_1)$. Then by Lemma 3.2 in [2] E has the same homotopy type as X.

References

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