17. *An Application of the Large Sieve*

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1. Let k be a positive integer and l an integer with $(l, k)=1, 0\leqq l<k$. We denote by $p(k, l)$ the least prime number in the arithmetical progression $kn+l$ $(n=0, 1, 2, \cdots)$. In 1936 P. Turán [2] proved under the assumption of the extended Riemann hypotheses for Dirichlet's L-functions that we have for any fixed number $\varepsilon>0$

$$p(k, l)<\phi(k)(\log k)^{2+\varepsilon}$$

except possibly for $o(\phi(k))$ values of l, $(l, k)=1$, as $k\to\infty$, where $\phi(k)$ is the Euler totient function.

The purpose of this note is to present the following theorem, which is slightly weaker than the above result of Turán's but can be proved without any unproven hypothesis.

Theorem 1. *Let A be an arbitrary real number greater than* 3 *and ε be any number with $0<\varepsilon<A-3$. Then, for almost all positive integers k we have*

$$p(k, l)<\phi(k)(\log k)^{A}$$

except possibly for $\phi(k)(\log k)^{-\varepsilon}$ values of l with $(l, k)=1$, $1\leqq l<k$.

Here, we say that a statement is valid for almost all positive integers k, when the number of positive integers $k\leqq x$ without the stated property is of $o(x)$ for $x\to\infty$.

2. Our Theorem 1 is a direct consequence of some large sieve result, which will be formulated as follows.

For integers k, l with $k\geqq 1, (k, l)=1$, we write as usual

$$\theta(x, k, l)=\sum_{\substack{p\leqq x\\ p\equiv 0(\mathrm{mod}\ k)}}\log p,$$

where $x(>1)$ is a real variable and where under the summation symbol $\sum p$ runs through over the prime numbers satisfying the conditions imposed there.

Theorem 2. *Let $A>0$ be any fixed number. Then, if*

$$Q\leqq x(\log x)^{-A},$$

we have

$$\sum_{k\leqq Q}\sum_{\substack{l\\(l,k)=1}}\sup_{y\leqq x}\left(\theta(y, k, l)-\frac{y}{\phi(k)}\right)^{2}\leqq B\frac{x^{2}}{(\log x)^{A-3}}$$

with a constant $B=B(A)>0$.

This result may be obtained by applying the large sieve method, initiated by Ju. V. Linnik and developed by A. Rényi, K. F. Roth,

E. Bombieri and others (cf. e.g. [1]), especially to certain maximal functions appropriately designed; the process that leads to results of this sort might be called the *maximal* large sieve.

In the next section we shall give a proof of Theorem 1 on the basis of Theorem 2. The proof for Theorem 2 will be published elsewhere.

3. Suppose now that $A>3$ and $0<\varepsilon<A-3$, and take a δ satisfying $0<\delta<(A-3-\varepsilon)/2$. Put

$$Q=\frac{x}{(\log x)^A},\ Q_0=\frac{x}{(\log x)^{A+\delta}}.$$

Since we have $\theta(y,k,l)=0$ for $y<p(k,l)$, it easily follows from the inequality in Theorem 2 that

$$\sum_{k\leqq Q}\sum_{\substack{l\\(l,k)=1}}\left(\frac{\min(p(k,l),x)}{\phi(k)}\right)^2\leqq B\frac{x^2}{(\log x)^{A-3}}. \tag{3}$$

We consider the number N_1 of positive integers $k\leqq Q$ such that (2): $p(k,l)\geqq x$ for more than

$$\frac{\phi(k)}{2(\log k)^\varepsilon}$$

values of l with $(l,k)=1, 1\leqq l<k$. Indicating by $\sum'$ the summation over positive integers $k\leqq Q$ satisfying the condition (2), we get, by (1),

$$\sum_{k\leqq Q}{}' \frac{\phi(k)}{2(\log k)^\varepsilon}\left(\frac{x}{\phi(k)}\right)^2\leqq B\frac{x^2}{(\log x)^{A-3}}$$

so that

$$N_1\frac{x^2}{2(\log Q)^\varepsilon Q}\leqq B\frac{x^2}{(\log x)^{A-3}},$$

whence

$$N_1\leqq B\frac{Q}{(\log x)^{A-3-\varepsilon}}=o(Q)$$

for $x\to\infty$. Hence, if we remove all such integers k from the interval

$$Q_0<k\leqq Q,$$

then the number N_2 of remaining integers k in this interval for which $p(k,l)\geqq\phi(k)(\log k)^A$ holds for more than

$$\left(1-\frac{1}{2(\log k)^\varepsilon}\right)\frac{\phi(k)}{2(\log k)^\varepsilon}$$

values of l with $(l,k)=1, 1\leqq l<k$, satisfies, by (1) again,

$$N_2\frac{Q_0(\log Q_0)^{2A}}{2(\log Q)^\varepsilon(\log Q_0)^\delta}\leqq B\frac{x^2}{(\log x)^{A-3}}$$

or

$$N_2\leqq B\frac{x}{(\log x)^{2A-3-\varepsilon-2\delta}}=o(Q)$$

for $x\to\infty$. Since $Q_0=o(Q)$ $(x\to\infty)$, we thus conclude that for all positive integers $k\leqq Q$, possibly with $N_1+N_2+[Q_0]=o(Q)$ exceptions, we have

$$p(k, l) > \phi(k) (\log k)^A$$

for at least

$$\left(1 - \frac{1}{2(\log k)^\varepsilon}\right)^2 \phi(k) > \left(1 - \frac{1}{(\log k)^\varepsilon}\right) \phi(k)$$

values of l with $(l, k) = 1, 1 \leqq l < k$. This proves our Theorem 1, since Q tends to infinity with x.

References

[1] P. X. Gallagher: The large sieve. Mathematika, **14**, 14–20 (1967).
[2] P. Turán: Über die Primzahlen der arithmetischen Progression. Acta Sci. Math. Szeged, **8**, 226–235 (1936/37).