42. L-ideals of Measure Algebras

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1. Introduction. Let G be a non-discrete locally comapct abelian group with the dual group Γ of G. We will denote by $M(G)$ the Banach algebra of all bounded regular Borel measures on G under convolution multiplication. If $\mu, \nu \in M(G)$, then their convolution product will be denoted $\mu * \nu$. We shall use additive notation for the group operation in G.

If $\mu, \nu \in M(G)$, then "$\nu \ll \mu$" will mean "ν is absolutely continuous with respect to μ" and "$\mu \perp \nu$" will mean "μ and ν are mutually singular". If $\mathfrak{M}$ is a closed subspace (subalgebra, ideal) of $M(G)$ will be called an *L-subspace* (*L-subalgebra*, *L-ideal*) provided $\mu \in \mathfrak{M}, \nu \in M(G)$ and $\nu \ll \mu$ imply $\nu \in \mathfrak{M}$. If $\mathfrak{M}$ is an L-subspace and $\mu \in M(G)$, then we say $\mu \perp \mathfrak{M}$ provided $\mu \perp \nu$ for each $\nu \in \mathfrak{M}$. We set $\mathfrak{M}^{\perp} = \{\mu \in M(G) : \mu \perp \mathfrak{M}\}$.

It is known that *there exists a compact commutative topological semigroup S with identity and an order preserving isometric isomorphism θ of* $M(G)$ into $M(S)$ such that:

T-(a) *the image of $M(G)$ in $M(S)$ is weak-$*$ dense*:

T-(b) *each multiplicative linear functional h on $M(G)$ has the form $h(\mu) = \int_S f d\theta\mu$ for some non-zero continuous semicharacter on S*;

T-(c) *there are enough non-zero continuous semicharacter on S to separate points*; and

T-(d) *if $\mu \in M(G)$, $\nu \in M(S)$ and $\nu \ll \theta\mu$ then there is a measure $\omega \in M(G)$ such that $\omega \ll \mu$ and $\theta\omega = \nu$* (cf. [2]).

We call S the *structure semigroup* of $M(G)$. The space of all non-zero continuous semicharacters on S is denoted by $\hat{S}$. We may consider $\hat{S}$ to be the maximal ideal space of $M(G)$, if we define the Gelfand transform of $\mu \in M(G)$ by $\hat{\mu}(f) = \int_S f d\theta\mu$ for $f \in \hat{S}$, and give $\hat{S}$ the weakest topology under which all of the functions $\hat{\mu}$ for $\mu \in M(G)$ are continuous. Since $M(G)$ has identity, $\hat{S}$ is a compact semigroup under pointwise multiplication. Pointwise multiplication is not generally continuous in the Gelfand topology. However, for fixed $g \in \hat{S}$ it is easily seen that the map $f \to gf$ is weakly continuous. We may consider Γ to be the maximal group at identity. In other word, $\Gamma = \{f \in S : |f| \equiv 1\}$. As well known, if $\mu \in M(G)$ and $\hat{\mu}(f) = 0$ for all $f \in \Gamma$, then $\mu = 0$.

We denote by $\varDelta$ the subset of $\hat{S}$ consisting of functionals symmereic in the sense that $\hat{\mu}^*(f)=\overline{\hat{\mu}(f)}$ for any $\mu\in M(G)$, where $*$ denotes the usual involution on $M(G)$. Let $\mathfrak{N}(\hat{S}\backslash\varDelta)=\{\mu\in M(G):\hat{\mu}(f)=0$ for all $f\in\hat{S}\backslash\varDelta\}$. J. H. Williamson showed the following result ([4]). "*Suppose $\mu\in\mathfrak{N}(\hat{S}\backslash\varDelta)$ and $\mu=\mu_1+\mu_2$, where μ_1 is atomic and μ_2 continuous. Then* $\sup_{f\in\hat{S}}|\hat{\mu}_1(f)|<\sup_{f\in\hat{S}}|\hat{\mu}_2(f)|$."

The main purpose of this paper is to show that *if $\mu\in\mathfrak{N}(\hat{S}\backslash\varDelta)$, then μ is a continuous measure of $M(G)$.*

We give some preliminaries in § 2. In § 3, we investigate L-ideals of $M(G)$. In § 4, we prove, using the result of § 3, that $\mathfrak{N}(\hat{S}\backslash\varDelta)$ is an L-ideal of $M(G)$, in particular $\mathfrak{N}(\hat{S}\backslash\varDelta)\subset M_C(G)$, where $M_C(G)$ is an L-ideal of all continuous measures on G.

2. Preliminaries. The following proposition follows directly from the Lebesgue decomposition theorem.

Proposition 1. *If $\mathfrak{M}$ is an L-subspace of $M(G)$, then so is $\mathfrak{M}^\perp$ and $M(G)=\mathfrak{M}\oplus\mathfrak{M}^\perp$.*

Let $\mathfrak{M}$ be an L-ideal of $M(G)$ which is not contained in $M_C(G)$. Since $\mathfrak{M}$ is an L-ideal, there is an element x of G such that δ_x, where δ_x is a unit mass concentrated at a point x, is an element of $\mathfrak{M}$. From that $\mathfrak{M}$ is an L-ideal of $M(G)$, $\delta_0=\delta_x*\delta_{-x}$ is an element of $\mathfrak{M}$. Thus, $\mathfrak{M}=M(G)$. Hence, we have the following proposition.

Proposition 2. *Every proper L-ideals of $M(G)$ are contained in $M_C(G)$.*

Definition 1. If $\mathfrak{M}$ is an L-ideal of $M(G)$ and $\mathfrak{M}^\perp$ is a subalgebra, then $\mathfrak{M}$ will be called a prime L-ideal.

Definition 2. An ideal J of S, such that $S\backslash J$ is a subsemigroup of S, will be called a prime ideal.

For $f\in\hat{S}$, let $J(f)=\{s\in S: f(s)=0\}$, then $J(f)$ is a prime ideal of S. Put $\mathfrak{N}(f)=\{\mu\in M(G):\theta\mu$ is concentrated on $J(f)\}$, then $\mathfrak{N}(f)$ is a prime L-ideal of $M(G)$.

The following theorem is showed by J. L. Taylor.

Theorem 1 (J. L. Taylor [2]). *If $\mathfrak{M}$ is a proper L-subspace of $M(G)$, then the following statements are equivalent:*

(a) *$\mathfrak{M}$ is a prime L-ideal;*

(b) *there is an idempotent semicharacter $\pi\in\hat{S}$ such that* $\mathfrak{M}=\left\{\mu\in M(G):\int_S\pi d\theta\,|\mu|=0\right\}$;

(c) *there is a semicharacter $f\in\hat{S}$ such that $\mathfrak{M}=\mathfrak{N}(f)$;*

(d) *there is an open compact prime ideal J of S such that* $\mathfrak{M}=\{\mu\in M(G):\theta\mu$ *is concentrated on* $J\}$.

The following proposition follows from T-(d) in § 1.

Proposition 3. *If $\mu\in M(G)$ and $g\in\hat{S}$, then there is a measure $\mu_g\in M(G)$ such that $d\theta\mu_g=gd\theta\mu$.*

3. L-ideals. Definition 3. A subset Λ of $\hat{S}$, such that $f\cdot\Lambda\subset\Lambda$ for every $f\in\Gamma$, will be called a *Γ-invariant set.*

Theorem 2. *Let $\mathfrak{M}$ be an L-subspace of $M(G)$. If $\Lambda(\mathfrak{M})=\{f\in\hat{S}:\hat{\mu}(f)=0$ for all $\mu\in\mathfrak{M}\}$, then $\Lambda(\mathfrak{M})$ is a closed ideal of $\hat{S}$.*

Proof. It is obvious that $\Lambda(\mathfrak{M})$ is closed. Since $\mathfrak{M}$ is an L-subspace, if $g\in\hat{S}$ and $\mu\in\mathfrak{M}$, then $\mu_g\in\mathfrak{M}$. Thus, if $g\in\hat{S}$ and $f\in\Lambda(\mathfrak{M})$, then

$$\int_S fg\,d\theta\mu=\int_S f\,d\theta\mu_g=0$$

for all $\mu\in\mathfrak{M}$. It follows that $fg\in\Lambda(\mathfrak{M})$. Thus, $\Lambda(\mathfrak{M})$ is a closed ideal of $\hat{S}$. The theorem is proved.

For $f\in\hat{S}$, let $S(f)=S\backslash J(f)$, and let $(\theta\mu)_{S(f)}$ be the restriction to $S(f)$ of $\theta\mu$ for $\mu\in M(G)$. If $g\in\Gamma$, then

$$\int_S g\,d\theta\mu_f=\int_S gf\,d\theta\mu=\int_S gf\,d(\theta\mu)_{S(f)}.$$

Thus, we have the following lemma.

Lemma 1. *If $\mu\in M(G)$ and $f\in\hat{S}$, then $\mu_f=0$ if and only if $(\theta\mu)_{S(f)}=0$.*

For any subset Λ of $\hat{S}$, we set $\mathfrak{N}(\Lambda)=\{\mu\in M(G):\hat{\mu}(f)=0$ for every $f\in\Lambda\}$.

Theorem 3. *If Λ is a Γ-invariant subset of $\hat{S}$, then $\mathfrak{N}(\Lambda)$ is an L-ideal of $M(G)$. In particular, if Λ is non-empty, then $\mathfrak{N}(\Lambda)\subset M_c(G)$.*

Proof. Since Λ is Γ-invariant, if $f\in\Lambda$ and $\mu\in\mathfrak{N}(\Lambda)$, then

$$\int_S g\,d\theta\mu_f=\int_S gf\,d\theta\mu=0$$

for all $g\in\Gamma$. Thus, from the uniqueness of Fourier-Stieltjes transform, $\theta\mu_f=0$. It follows from Lemma 1 that $\theta\mu$ is concentrated on $J(f)$ for all $f\in\Lambda$. Hence, if we put $J(\Lambda)=\bigcap_{f\in\Lambda}J(f)$ and $\mathfrak{M}(J(\Lambda))=\{\mu\in M(G):\theta\mu$ is concentrated on $J(\Lambda)\}$, then $\mathfrak{N}(\Lambda)\subset\mathfrak{M}(J(\Lambda))$. Conversely, if $\mu\in\mathfrak{M}(J(\Lambda))$, then

$$\hat{\mu}(f)=\int_S f\,d\theta\mu=0$$

for all $f\in\Lambda$, Thus, $\mu\in\mathfrak{N}(\Lambda)$. Hence, it follows that $\mathfrak{N}(\Lambda)=\mathfrak{M}(J(\Lambda))$. Futhermore, since $\mathfrak{M}(J(\Lambda))$ is an intersection of prime L-ideals, $\mathfrak{N}(\Lambda)$ is an L-ideal of $M(G)$. Since a measure $\omega\in\theta(M_d(G))$, where $M_d(G)$ is the subspace of all discrete measures on G, is concentrated on $S(f)$ for any $f\in\hat{S}$ ([3]), from Proposition 2, if Λ is non-empty, then $\mathfrak{N}(\Lambda)\subset M_c(G)$. This completes the proof.

Corollary. *A measure μ on G is continuous if and only if $\hat{\mu}$ vanishes on some non-empty Γ-invariant subset of $\hat{S}$.*

Corollary. *Let Λ be a Γ-invariant subset of $\hat{S}$. If $[\Lambda]$ is a smallest closed ideal of $\hat{S}$ which contains Λ, then $\mathfrak{N}(\Lambda)=\mathfrak{N}([\Lambda])$, in other word, if $\mu\in\mathfrak{N}(\Lambda)$, then $\hat{\mu}(f)=0$ for every $f\in[\Lambda]$.*

4. Application. (1) Let $M_0(G)$ be the subalgebra of $M(G)$ consisting of all measures whose Fourier transform vanishes on $\bar{\Gamma}\backslash\Gamma$. In view of that for fixed $g\in\hat{S}$ the map $f\to gf$ is continuous, $\bar{\Gamma}\backslash\Gamma$ is Γ-invariant. Thus, the next theorem is followed.

Theorem 5. *$M_0(G)$ is an L-ideal of $M(G)$.*

(2) From now, we shall investigate the subalgebra $\mathfrak{N}(\hat{S}\backslash\Delta)$.

Lemma 2. *If Λ is a Γ-invariant subset of $\hat{S}$, then so is $\hat{S}\backslash\Lambda$.*

Proof. Suppose that there is a semicharacter $g\in\hat{S}\backslash\Lambda$ such that $fg\in\Lambda$ for some $f\in\Gamma$. Since $\bar{f}\in\Gamma$ and $|f|\equiv1$, we have that $\bar{f}fg=g\in\Lambda$. This is impossible. Thus, $\hat{S}\backslash\Lambda$ is Γ-invariant. This completes the proof.

Lemma 3. *Δ is a Γ-invariant set of $\hat{S}$.*

Proof. At first, we shall show that if $f\in\Gamma$ and $\mu\in M(G)$, then $d\theta(\mu_f)^*=fd\theta\mu^*$. Since $gf\in\Gamma$ for every $g\in\Gamma$,

$$\int_S gd\theta(\mu_f)^*=\overline{\int_S gd\theta\mu_f}=\overline{\int_S gfd\theta\mu}=\int_S gfd\theta\mu^*.$$

Thus, in view of the uniqueness of Fourier-Stieltjes transform, $d\theta(\mu_f)^*=fd\theta\mu^*$. If $f\in\Gamma$ and $g\in\Delta$, then

$$\int_S fgd\theta\mu^*=\int_S gd\theta(\mu_f)^*=\overline{\int_S gd\theta\mu_f}=\overline{\int_S gfd\theta\mu}.$$

Thus, $fg\in\Delta$. This completes the proof.

Theorem 4. *$\mathfrak{N}(\hat{S}\backslash\Delta)$ is an L-ideal of $M(G)$. In particular, $\mathfrak{N}(\hat{S}\backslash\Delta)\subset M_c(G)$.*

Proof. From Lemma 2 and Lemma 3, $\hat{S}\backslash\Delta$ is Γ-invariant. Thus, $\mathfrak{N}(\hat{S}\backslash\Delta)$ is an L-ideal of $M(G)$. As well known, since G is non-discrete, $\hat{S}\backslash\Delta$ is non-empty. Thus, from Theorem 3, $\mathfrak{N}(\hat{S}\backslash\Delta)\subset M_c(G)$. The theorem is proved.

If $\mathfrak{M}$ is a subset of $M(G)$ such that $\mu\in\mathfrak{M}$ implies $\mu^*\in\mathfrak{M}$, then $\mathfrak{M}$ will be called *symmetric*.

Lemma 4. *If π is an idempotent of $\hat{S}$, then the following statements are equivalent:*

(a) *π is an element of Δ;*

(b) *$\mathfrak{N}(\pi)$ is a symmetric prime L-ideal of $M(G)$.*

Proof. Suppose that there is a measure $\mu\in\mathfrak{N}(\pi)$ such that $\mu^*\notin\mathfrak{N}(\pi)$. Then, $(\theta\mu^*)_{S(\pi)}$ is a non-zero measure of $M(S)$. From T-(d), there is a measure $\omega\in M(G)$ such that $\theta\omega^*=(\theta\mu^*)_{S(\pi)}$ and $\omega\ll\mu$. Then, it follows that

$$|\hat{\omega}^*|(\pi)=\int_S \pi d\theta|\omega^*|=|\omega^*|(G)>0.$$

On the other hand, since $\omega\ll\mu$, $|\omega|\in\mathfrak{N}(\pi)$. It follows that

$$|\hat{\omega}|(\pi)=\int_S \pi d\theta|\omega|=0.$$

Therefore, $\pi\notin\Delta$. Hence, (a) implies (b). Suppose that $\mathfrak{N}(\pi)$ is symmetric. For any $\mu \in M(G)$, let $\mu=\mu_1+\mu_2$, where $\mu_1 \in \mathfrak{N}(\pi)$ and $\mu_2 \in \mathfrak{N}(\pi)^\perp$. Since $\mathfrak{N}(\pi)^\perp$ is also symmetric, we have that

$$\hat{\mu}^*(\pi)=\int_S \pi d\theta\mu^*=\int_{S(\pi)} d\theta\mu^*=\mu_2^*(G)$$
$$=\overline{\mu_2(G)}=\overline{\int_{S(\pi)} d\theta\mu_2}=\overline{\int_S \pi d\theta\mu}=\overline{\hat{\mu}(\pi)}.$$

Thus, $\pi \in \Delta$. Hence, (b) implies (a). This completes the proof.

The following theorem follows directly from this lemma.

Theorem 5. *If $\mathfrak{M}$ is a non-symmetric prime L-ideal of $M(G)$, then $\mathfrak{N}(\hat{S}\backslash\Delta)\subset\mathfrak{M}$.*

If $\mathfrak{M}$ is a non-symmetric prime L-ideal of $M(G)$, then so is $\mathfrak{M}^*$, where $\mathfrak{M}^*=\{\mu^* \in M(G): \mu \in \mathfrak{M}\}$. Thus, if $\mathfrak{M}$ is a prime L-ideal of $M(G)$ such that $\mathfrak{N}(\hat{S}\backslash\Delta)\subset\mathfrak{M}$, then $\mathfrak{N}(\hat{S}\backslash\Delta)\subset\mathfrak{M}^*$. Thus, we have the following fact as the corollary to Theorem 5.

Corollary. *If $\mu \in \mathfrak{N}(\hat{S}\backslash\Delta)$, then $\mu^* \in \mathfrak{N}(\hat{S}\backslash\Delta)$.*

Theorem 6. *Let H be a non-open closed subgroup of G. Let $\mathfrak{A}$ be a collection of all countable unions of cosets of H. If $M(\mathfrak{A})$ is a closed subalgebra of $M(G)$ consisting of all measures that are concentrated on $\mathfrak{A}$. Then, $\mathfrak{N}(\hat{S}\backslash\Delta)\subset M(\mathfrak{A})^\perp$.*

Proof. Let $\hat{S}_0$ be the maximal ideal space of $M(G/H)$. Since G/H is non-discrete, there is a non-symmetric multiplicative linear functional f_0 on $M(G/H)$. If Φ is a canonical homomorphism of $M(G)$ *onto* $M(G/H)$, then there is a continuous injection mapping α of $\hat{S}_0$ into $\hat{S}$ such that $\hat{\mu}(\alpha f)=\widehat{\Phi\mu}(f)$ for $f \in \hat{S}_0$ and $\mu \in M(G)$. Since Φ maps $M(G)$ onto $M(G/H)$, αf_0 is a non-symmetric multiplicative linear functional on $M(G)$. Suppose that $\mathfrak{N}(\hat{S}\backslash\Delta)\not\subset M(\mathfrak{A})^\perp$. Since $\mathfrak{N}(\hat{S}\backslash\Delta)$ is an L-ideal, $\mathfrak{N}(\hat{S}\backslash\Delta)\cap M(\mathfrak{A})\neq\{0\}$. Clearly, $\mathfrak{N}(\hat{S}\backslash\Delta)\cap M(\mathfrak{A})$ is an L-ideal of $M(G)$. Thus, there is a positive measure μ_0 of $\mathfrak{N}(\hat{S}\backslash\Delta)$ with norm 1 whose support lies in H. Then, it follows that $\Phi(\mu_0)$ is identity of $M(G/H)$. Therefore, $\hat{\mu}_0(\alpha f_0)=\widehat{\Phi\mu_0}(f_0)=1$. This is impossible. Thus $\mathfrak{N}(\hat{S}\backslash\Delta)\subset M(\mathfrak{A})^\perp$. The theorem is proved.

References

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