52. The Theory of Nuclear Spaces Treated by the Method of Ranked Space. VI

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(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1972)

In the paper [8], we have studied the dual space of the extended nuclear space. In this paper we shall continue to do it.

§ 7. The dual space. (2).

Lemma 39. (1) *$V^*(0,h,i)$ is circled.*

(2) *$V^*(0,h,i)+V^*(0,k,j)=V^*(0,[hk/h+k],\min(i,j))$ for $h,k>1$.*

Proof. (1) It is clear.

(2) Suppose $i<j$. Then we have

$$V^*(0,h,i)+V^*(0,k,j)\subseteqq V^*(0,h,j)+V^*(0,k,j)$$

by Lemma 37 in [8]. Now, let F_1 and F_2 belong to $V^*(0,h,j)$ and $V^*(0,k,j)$, respectively. Then we have $|F_1(g)|<\varepsilon_j/h$ and $|F_2(g)|<\varepsilon_j/k$ to every $g\in\hat{V}_j(0,1,j)$, hence we obtain $|F_1(g)+F_2(g)|\leqq|F_1(g)|+|F_2(g)|<\varepsilon_j(h+k)/hk<\varepsilon_j/l$, where $l=[hk/h+k]$. This proof is complete. The sequence of neighbourhoods, $\{V^*(0,\gamma(h),i(h))\}$, where

$$V^*(0,\gamma(h),i(h))\supseteqq V^*(0,\gamma(h+1),i(h+1)),\ \gamma(h)\leqq\gamma(h+1)$$

and $\gamma(h)\to\infty$ as $h\to\infty$, is a fundamental sequence of neighbourhoods in $\hat{\Phi}'$.

Lemma 40. *If $\{V^*(0,\gamma(h),i(h))\}$ is a fundamental sequence of neighbourhoods in $\hat{\Phi}'$, then $F\in V^*(0,\gamma(h),i(h))$ for every integer h implies $F=0$, that is, $F(g)=0$ for every $g\in\hat{\Phi}$.*

Proof. By Lemma 38 in [8], we have $\min_h\{i(h)\}\geqq 1$. We write briefly $\min_h\{i(h)\}=j$. Hence there exists some integer N such that the relation $h\geqq N$ implies $i(h)=j$. The fact that F belongs to $V^*(0,\gamma(h),j)$ for $h\geqq N$ follows $F\in M_j^0$ and $|F(g)|<\varepsilon_j/\gamma(h)$ for $g\in\hat{V}_j(0,1,j)$. And since $g/2\hat{P}_j(g)$ belongs to $\hat{V}_j(0,1,j)$ for any element $g\in\hat{\Phi}$ with $P_j(g)\neq 0$, we see $|F(g)/2\hat{P}_j(g)|<\varepsilon_j/\gamma(h)$. Consequently we obtain

$$|F(g)|<2\varepsilon_j\hat{P}_j(g)/\gamma(h).$$

That shows $F(g)=0$ for every $g\in\hat{\Phi}$. This proof is complete.

Now, we can prove that the linear space $\hat{\Phi}'$ is a linear ranked space, by M. Washihara, [3].

Theorem 7. *The linear ranked space $\hat{\Phi}'$ is complete with respect to the R-convergence.*

Proof. Let $\{F_n\}$ be an R-cauchy sequence of elements in $\hat{\Phi}'$. Then there exists some fundamental sequence of neighbourhoods

$$\{V^*(0,\gamma(h),i(h))\}$$

such that the relations $n \geqq h$ and $m \geqq h$ imply $F_n - F_m \in V^*(0, \gamma(h), i(h))$. When we write briefly $\min_h \{i(h)\} = j$, there exists some integer N such that the relations $h \geqq N$ implies $i(h) = j$. Hence we have

$$F_n - F_m \in V^*(0, \gamma(h), j)$$

to $n, m \geqq h \geqq N$, that is, $|F_n(g) - F_m(g)| < \varepsilon_j / \gamma(h)$ to every $g \in \hat{V}_j(0, 1, j)$, and $F_n - F_m \in M_j^0$. Thus the sequence of numbers $\{F_n(g)\}$ has a limit number depending on $g \in \hat{V}_j(0, 1, j)$. For all $g \in \hat{\Phi}$, with $\hat{P}_j(g) \neq 0$ we have $g / 2\hat{P}_j(g) \in \hat{V}_j(0, 1, j)$, so that we obtain a linear functional

$$F(g / 2\hat{P}_j(g)) = \lim_{n \to \infty} F_n(g / 2\hat{P}_j(g)), \text{ i.e., } F(g) = \lim_{n \to \infty} F_n(g).$$

Hence we have $F(g) = \lim_{n \to \infty} F_n(g)$ for all $g \in \hat{\Phi}$. Then there exists some integer l such that $F_n \in M_l^0$ for all $n \geqq N$ and $F \in M_l^0$.

Next, we shall prove that $F(g)$ is R-continuous, that is, $F(g_\xi) \to F(g)$ as $g_\xi \xrightarrow{R} g$. We have

$$\begin{aligned} |F(g) - F(g_\xi)| &= \left| \sum_{k=1}^{l} \lambda_{k, n_{l-1}, n_l} (g - g_\xi, \varphi_{k, n_l})_{n_l} F(\varphi_{k, n_{l-1}}) \right| \\ &\leqq \varepsilon_l \left(\sum_{k=1}^{l} (\lambda_{k, n_{l-1}, n_l} / \varepsilon_l)^2 |(g - g_\xi, \varphi_{k, n_l})_{n_l}|^2 \right)^{1/2} \left(\sum_{k=1}^{l} |F(\varphi_{k, n_{l-1}})|^2 \right)^{1/2} \\ &\leqq \varepsilon_l \hat{P}_l(g - g_\xi) \left(\sum_{k=1}^{l} |F(\varphi_{k, n_{l-1}})|^2 \right)^{1/2}. \end{aligned}$$

Theorem 8. *A linear functional F belongs to $V^*(0, h, i)$ if and only if $F \in M_i^0$ and $(\sum_{k=1}^{i} |F(\varphi_{k, n_{i-1}})|^2)^{1/2} \leqq 1/h$.*

Proof. Suppose $F \in M_i^0$ and $(\sum_{k=1}^{i} |F(\varphi_{k, n_{i-1}})|^2)^{1/2} \leqq 1/h$. To any $g \in \hat{V}_i(0, 1, i)$ we have

$$\begin{aligned} |F(g)| &= \left| \sum_{k=1}^{i} \lambda_{k, n_{i-1}, n_i} (g, \varphi_{k, n_i})_{n_i} F(\varphi_{k, n_{i-1}}) \right| \\ &\leqq \left(\sum_{k=1}^{i} (\lambda_{k, n_{i-1}, n_i} / \varepsilon_i)^2 |(g, \varphi_{k, n_i})_{n_i}|^2 \right)^{1/2} \varepsilon_i \left(\sum_{k=1}^{i} |F(\varphi_{k, n_{i-1}})|^2 \right)^{1/2} < \varepsilon_i / h, \end{aligned}$$

then F belongs to $V^*(0, h, i)$.

Next, suppose F belongs to $V^*(0, h, i)$. This means $F \in M_i^0$ and $|F(g)| < \varepsilon_i / h$ for all $g \in \hat{V}_i(0, 1, i)$, that is, $|F(g)| \leqq \varepsilon_i \hat{P}_i(g) / h$.

On the other hand, since we have

$$\begin{aligned} \hat{P}_i(g) &= \left\| \sum_{k=1}^{i} (\lambda_{k, n_{i-1}, n_i} / \varepsilon_i)(g, \varphi_{k, n_i})_{n_i} \varphi_{k, n_{i-1}} \right\|_{n_{i-1}} \\ &= \left(\sum_{k=1}^{i} (\lambda_{k, n_{i-1}, n_i} / \varepsilon_i)^2 |(g, \varphi_{k, n_i})_{n_i}|^2 \right)^{1/2} \end{aligned}$$

and

$$F(g) = \sum_{k=1}^{i} \lambda_{k, n_{i-1}, n_i} (g, \varphi_{k, n_i})_{n_i} F(\varphi_{k, n_{i-1}}),$$

we obtain

$$h \left| \sum_{k=1}^{i} \lambda_{k, n_{i-1}, n_i} (g, \varphi_{k, n_i})_{n_i} F(\varphi_{k, n_{i-1}}) \right| \leqq \left(\sum_{k=1}^{i} (\lambda_{k, n_{i-1}, n_i})^2 |(g, \varphi_{k, n_i})|^2 \right)^{1/2} \quad (1)$$

In particular, we set

$$g = \sum_{k=1}^{i} (\lambda_{k, n_{i-1}, n_i})^{-1} \overline{F(\varphi_{k, n_{i-1}})} \varphi_{k, n_i} \qquad \text{in the equation (1),}$$

then we have

$$\left(\sum_{k=1}^{i}|F(\varphi_{k,n_{i-1}})|^2\right)^{1/2}\leqq 1/h.$$

§8. The second dual space.

Definition 12. We say that a linear functional $\mathfrak{F}$ defined on the linear ranked space $\hat{\Phi}'$ is R-continuous, if we have $\lim_{n\to\infty}\mathfrak{F}(F_n)=\mathfrak{F}(F)$ to any R-convergence sequence $\{F_n\}$ such that $F_n\xrightarrow{R}F$ in $\hat{\Phi}'$. Furthermore let $\hat{\Phi}''$ be the set of all R-continuous linear functionals on $\hat{\Phi}'$. We call it the second dual space.

Definition 13. We define

$$V_i^{**}(0,r,i)=\{\mathfrak{F}\in\hat{\Phi}'';|\mathfrak{F}(F)|<\varepsilon_i r \text{ for all } F\in V^*(0,1,i)\},$$

where r is a positive number, as a neighbourhood of the origin in $\hat{\Phi}''$. We denote briefly $V_i^{**}(0)\equiv V_i^{**}(0,1/i,i)$ and call it a neighbourhood of the origin with rank i.

Furthermore we define that the neighbourhood with rank 0, V_i^{**} is always the space $\hat{\Phi}''$.

Lemma 41. *We have* $V_j^{**}(0,1,j)\supseteqq V_i^{**}(0,1,i)$ *if* $j\leqq i$.

Proof. If $j\leqq i$, we have $V^*(0,1,j)\subseteqq V^*(0,1,i)$ by Lemma 37, and $\varepsilon_j\geqq\varepsilon_i$. Hence if $\mathfrak{F}$ belongs to $V_i^{**}(0,1,i)$, we obtain $|\mathfrak{F}(F)|<\varepsilon_i\leqq\varepsilon_j$ to all $F\in V^*(0,1,j)$.

Lemma 42. *We have* $V_j^{**}(0)\supseteqq V_i^{**}(0)$ *if* $j\leqq i$.

Proof. It is clear from

$$V_i^{**}(0)\equiv V_i^{**}(0,1/i,i)\subseteqq V_j^{**}(0,1/i,j)\subseteqq V_j^{**}(0,1/j,j)\equiv V_j^{**}(0).$$

Lemma 43. (1) *$V_i^{**}(0)$ is circled.*

(2) *To* $i,j>1$, $V_i^{**}(0)+V_j^{**}(0)\subseteqq V_k^{**}(0)$ *with* $k=\left[\dfrac{\min(i,j)}{2}\right]$.

Proof. (1) It is evident.

(2) Suppose $j\leqq i$. Then we have

$$V_i^{**}(0)+V_j^{**}(0)\subseteqq V_j^{**}(0)+V_j^{**}(0)\equiv V_j^{**}(0,1/j,j)+V_j^{**}(0,1/j,j)$$
$$\subseteqq V_j^{**}(0,2/j,j)\subseteqq V_{[j/2]}^{**}(0,1/[j/2],[j/2])=V_{[j/2]}^{**}(0).$$

Hence we obtain (2). Q.E.D.

Thus we see by M. Washihara, [3] that the linear space $\hat{\Phi}''$ is the linear ranked space, and the sequence of neighbourhoods, $\{V_{\gamma(i)}^{**}(0)\}$ with $\gamma(i)\leqq\gamma(i+1)$ and $\gamma(i)\to\infty$, is the fundamental sequence.

Lemma 44. *If* $\{V_{\gamma(i)}^{**}(0)\}$ *is a fundamental sequence of neighbourhoods in* $\hat{\Phi}''$, *then* $\mathfrak{F}\in V_{\gamma(i)}^{**}(0)$ *to every integer i implies* $\mathfrak{F}=0$, *that is,* $\mathfrak{F}(F)=0$ *to every* $F\in\hat{\Phi}'$.

Proof. Let F be any element in $\hat{\Phi}'$, then there exists some integer j such that $F\in M_j^0$. Theorem 8 leads

$$\left\{F\Big/\left(\sum_{k=1}^{j}|F(\varphi_{k,n_{j-1}})|^2\right)^{1/2}\right\}\in V^*(0,1,j).$$

Hence we have

$$\left\{F\Big/\left(\sum_{k=1}^{j}|F(\varphi_{k,n_{j-1}})|^2\right)^{1/2}\right\}\in V^*(0,1,\gamma(i))\qquad\text{for } \gamma(i)\geqq j.$$

Thus we obtain $|\mathfrak{F}(F)|<(\sum_{k=1}^{j}|F(\varphi_{k,n_{j-1}})|^2)^{1/2}\varepsilon_{\gamma(i)}/\gamma(i)$ for every $\gamma(i)\geqq j$. Since $\gamma(i)\to\infty$ as $i\to\infty$, we assert $\mathfrak{F}(F)=0$.

Theorem 9. *Let g and F belong to $\hat{\Phi}$ and $\hat{\Phi}'$ respectively, then $F(g)$ is a linear functional on $\hat{\Phi}'$. Furthermore $F(g)$ is R-continuous on $\hat{\Phi}'$.*

Proof. It is clear that $F(g)$ is a linear functional on $\hat{\Phi}'$, then we shall prove that $F(g)$ is R-continuous on $\hat{\Phi}'$, that is, $F_n(g)\to F(g)$ if $F_n\xrightarrow{R}F$ in $\hat{\Phi}'$.

Now, suppose $F_n\xrightarrow{R}F$ in $\hat{\Phi}'$, then there exists some fundamental sequence of neighbourhoods, $\{V^*(0,\gamma(h),i(h))\}$ such that the relation $n\geqq h$ implies $F_n-F\in V^*(0,\gamma(h),i(h))$. If we write briefly $\min_h\{i(h)\}=j$, there exists some integer N such that the relation $h\geqq N$ implies $i(h)=j$. Hence we have $F_n-F\in V^*(0,\gamma(h),j)$ for $n\geqq h\geqq N$.

Consequently for any element $g\in\hat{\Phi}$ such that $\hat{P}_j(g)\neq 0$, the relation $n\geqq h\geqq N$ implies $|(F_n-F)(g/2\hat{P}_j(g))|<\varepsilon_j/\gamma(h)$ and $F_n-F\in M_j^0$.

Since we have $F(g)=F_n(g)$ for an element $g\in\hat{\Phi}$ such that $\hat{P}_j(g)=0$, we assert $F_n(g)\to F(g)$ as $n\to\infty$.

Theorem 10. *By Theorem 9, the correspondence between $g\in\hat{\Phi}$ and $\mathfrak{F}\in\hat{\Phi}''$ defines a linear operator J on $\hat{\Phi}$ into $\hat{\Phi}''$. Then we have $R(J)=\hat{\Phi}''$, where $R(J)$ is the range of J.*

Proof. It is clear that J is a linear operator. Then we shall prove $R(J)=\hat{\Phi}''$. Let $\mathfrak{F}$ be an R-continuous linear functional defined on $\hat{\Phi}'=\bigcup_{i=1}^{\infty}M_i^0$ and $\mathfrak{F}_i$ be the restriction of $\mathfrak{F}$ to M_i^0.

Since $\mathfrak{F}$ is R-continuous, we have $\mathfrak{F}_i(F_n)\to\mathfrak{F}_i(F)$ if $F_n\xrightarrow{R}F$ with $F_n,F\in M_i^0$. On the other hand, $F_n\xrightarrow{R}F$ with $F_n,F\in M_i^0$ is equivalent to $\sum_{k=1}^{i}|(F_n-F)(\varphi_{k,n_{i-1}})|^2\to 0$ by Theorem 8. Since $(\sum_{k=1}^{i}|F(\varphi_{k,n_{i-1}})|^2)^{1/2}$ is a norm in the finite dimensional subspace M_i^0 by Lemma 35 in [8], $\mathfrak{F}_i$ is a continuous linear functional with respect to the norm on M_i^0.

By the paper [8], M_i^0 is the dual space of N_i, which is the finite dimensional subspace of $\hat{\Phi}$.

First, suppose $\mathfrak{F}_1$ is the restriction of $\mathfrak{F}$ to M_1^0. Then there exists some element g_1 in N_1 such that $\mathfrak{F}_1(F)=F(g_1)$ for all $F\in M_1^0$.

Second, suppose $\mathfrak{F}_2$ is the restriction of $\mathfrak{F}$ to M_2^0. Then we find some element g_2' in N_2 such that $\mathfrak{F}_2(F)=F(g_2')$ for all $F\in M_2^0$. By Lemma 28 in [8], N_1 is a subspace in N_2, so then there exists a subspace L_1 generated by φ_{2,n_1} such that $N_2=N_1\oplus L_1$. Thus we have $g_2'=g_1'+g_2$ such that $g_1'\in N_1$ and $g_2\in L_1$. Hence we have $\mathfrak{F}_2(F)=F(g_1'+g_2)=F(g_1')+F(g_2)$ for all $F\in M_2^0$. If $F\in M_1^0$, then $F\in M_2^0$. Then we obtain $F(g_1)=\mathfrak{F}_1(F)=\mathfrak{F}_2(F)=F(g_1')$ for all $F\in M_1^0$. Hence we have $g_1=g_1'$ in N_1, and then we obtain $g_2'=g_1+g_2$ such that $g_1\in N_1$ and $g_2\in L_1$. In the same manner, the restriction $\mathfrak{F}_i$ of $\mathfrak{F}$ to M_i^0 corresponds to some element g_i' in N_i such that $\mathfrak{F}_i(F)=F(g'_i)$ for all $F \in M_i^0$, and g'_i satisfies the following conditions,

(1) $g'_i=g_1+\cdots+g_i$,

(2) $g_1 \in N_1$ and $g_j \in L_{j-1}$, $j=2,\cdots,i$,

(3) $N_i=N_1 \oplus L_1 \oplus \cdots \oplus L_{i-1}$,

(4) L_j is a subspace generated by φ_{j+1,n_j}.

Thus the sequence $\{g'_i\}$ is an R-cauchy sequence of elements. Because, to any neighbourhood $\hat{V}_i(0,r,i)$ the relation $i<j$ implies $g'_j-g'_i \in \hat{V}_i(0,r,i)$, since $g'_j-g'_i=g_{i+1}+\cdots+g_j \in L_i \oplus \cdots \oplus L_{j-1} \subset M_i$.

Consequently there exists the limiting element of the sequence $\{\sum_{n=1}^i g_n\}_i$ in $\hat{\Phi}$. We denote it $\sum_{n=1}^\infty g_n$. Since to any element F in $\hat{\Phi}'$ there exists M_i^0 such that $F \in M_i^0$, we have

$$\mathfrak{F}(F)=\mathfrak{F}_i(F)=F(g'_i)=F\left(\sum_{n=1}^i g_n\right)=F\left(\sum_{n=1}^\infty g_n\right).$$

This proof is complete.

Theorem 11. *The correspondence $J(g)=\mathfrak{F}$ in Theorem 10 is bijective and we have $\mathfrak{F} \in V_i^{**}(0,r,i)$ if and only if $g \in \hat{V}_i(0,r,i)$.*

Proof. Let $\mathfrak{F}$ belong to $V_i^{**}(0,r,i)$ and g in $\hat{\Phi}$ be such that $J(g)=\mathfrak{F}$. Then we have $|F(g)|=|\mathfrak{F}(F)|<\varepsilon_i r$ for every $F \in V^*(0,1,i)$. Now we shall prove $g/r \in \hat{V}_i(0,1,i)$. Suppose it is not true, i.e., $g/r \notin \hat{V}_i(0,1,i)$. This means

$$\hat{P}_i(g/r)=\left\|\sum_{k=1}^i (\lambda_{k,n_{i-1},n_i}/\varepsilon_i)(g/r,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}} \geqq 1.$$

Put $A=(\sum_{k=1}^i (\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2|(g/r,\varphi_{k,n_i})_{n_i}|^2)^{1/2}$, then $A \geqq 1$.

We define a linear functional $F_0 \in M_i^0$ such that

$$\begin{cases} F_0(\varphi_{k,n_{i-1}})=\dfrac{1}{A}(\lambda_{k,n_{i-1},n_i}/\varepsilon_i)\,|(g/r,\varphi_{k,n_i})_{n_i}| & \text{for } k=1,\cdots,i, \\ F_0(\varphi_{k,n_{i-1}})=0 & \text{for } k>i. \end{cases}$$

Then we have

$$\left(\sum_{k=1}^i |F_0(\varphi_{k,n_{i-1}})|^2\right)^{1/2}=\frac{1}{A}\left(\sum_{k=1}^i (\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2|(g/r,\varphi_{k,n_i})_{n_i}|^2\right)^{1/2}=1.$$

Hence we obtain $F_0 \in V^*(0,1,i)$ by Theorem 8. On the other hand, we have by Lemma 36 in [8]

$$\begin{aligned} |F_0(g/r)| &= \left|\sum_{k=1}^i \lambda_{k,n_{i-1},n_i}(g/r,\varphi_{k,n_i})_{n_i}F_0(\varphi_{k,n_{i-1}})\right| \\ &=(\varepsilon_i/A)\sum_{k=1}^i (\lambda_{k,n_{i-1},n_i}/\varepsilon_i)^2|(g/r,\varphi_{k,n_i})_{n_i}|^2=\varepsilon_i A \geqq \varepsilon_i, \end{aligned}$$

that is, $|F_0(g)| \geqq \varepsilon_i r$ for $F_0 \in V^*(0,1,i)$. This is a contradiction. Next, we shall prove that $\mathfrak{F}=0$ implies $g=0$ for $J(g)=\mathfrak{F}$. If $\mathfrak{F}=0$, there exists a fundamental sequence of neighbourhoods $\{V_{\gamma(i)}^{**}(0)\}$ such that $\mathfrak{F} \in V_{\gamma(i)}^{**}(0)$ for all integer i. Hence g belongs to $\hat{V}_{\gamma(i)}(0) \equiv \hat{V}_{\gamma(i)}(0,1/\gamma(i),\gamma(i))$ for all integer i.

Since $\{\hat{V}_{\gamma(i)}(0)\}$ is a fundamental sequence of neighbourhoods in $\hat{\Phi}$, we have $g=0$. Thus the correspondence J is bijective. Finally, if

$g \in \hat{V}_i(0, r, i)$, we have $|F(g/r)| < \varepsilon_i$ for every $F \in V^*(0, 1, i)$. And then we obtain $|\mathfrak{F}(F)| = |F(g)| < \varepsilon_i r$ for $J(g) = \mathfrak{F}$.

Hence we have $\mathfrak{F} \in V_i^{**}(0, r, i)$.

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