69. A Note on the Dilation Theorems. II

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1. In the previous note [9], one of the authors discussed, jointly with Yamada, the mutual dependency of several dilation theorems. Especially, it is pointed out that Stinespring-Umegaki's algebra dilation theorem implies the so-called strong dilation theorem of Sz.-Nagy. However, the proofs of the implication are somewhat lengthy. In the present note, it will be shown that Stinespring-Umegaki's theorem can serve a proof of more general dilation theorem of Foiaş-Suciu [2]. Some consequences are also discussed.

2. The following theorem is the algebra dilation theorem due to [7] and [10]:

Theorem 1 (Stinespring-Umegaki). *If V is a completely positive (or positive definite) linear mapping defined on a unital C^*-algebra B with the range in the algebra $B(H)$ of all operators on a Hilbert space H, and V satisfies $V1=1$, then there is a ($*$-preserving) representation U of B on K such that*

(1) $$Vf=pUf|H$$

for any $f\in B$, where K includes H as a subspace and p is the projection of K onto H.

In the present note, the notion of the complete positivity is not necessary, since Stinespring [7; Theorem 4] established that the complete positivity coincides with the usual positivity if B is commutative which is the case treated in this note. Exactly, in the present note, B is always the algebra $C(X)$ of all continuous functions defined on a compact Hausdorff space X equipped with the sup-norm.

3. A subalgebra A of $C(X)$ is a *function algebra* on X if A satisfies

(i) A contains the constants, and

(ii) A separates the points of X.

A function algebra A is a *Dirichlet algebra* on X if the real part $\mathrm{Re}\,A$ of all real parts of functions belonging to A is dense in the algebra of all real continuous functions on X.

An *operator representation* V of a function algebra A on a Hilbert space H is an algebra homomorphism of A into $B(H)$ which satisfies

(2) $$V1=1$$

and

(3) $$\|Vf\| \leqq \|f\|,$$

for all $f \in A$. In the recent decade, the theory of operator representations is advanced by Foiaş, Mlak, Suciu and their colleagues. Following general theorem for Dirichlet algebras is proved in [2; Theorem 6]:

Theorem 2 (Foiaş-Suciu). *If V is an operator representation of a Dirichlet algebra A on H, then there is a ($*$-preserving) representation of $C(X)$ into $B(K)$ which satisfies* (1).

4. Comparing Theorem 2 with Theorem 1, one can easily deduce, the key of the present note lies in the fact that the operator representation V of A is extensible to a positive linear map W on $C(X)$; that is, the following diagram becomes commutative:

$$\begin{array}{ccc} C(X) & \xrightarrow{U} & B(K) \\ {\scriptstyle i}\big\uparrow & \searrow^{W} & \big\downarrow {\scriptstyle P=p\cdot|H} \\ A & \xrightarrow{V} & B(H). \end{array}$$

For the extension of V to $C(X)$, a natural task is to define

(5) $$W(\operatorname{Re} f) = \operatorname{Re} Vf.$$

By the cartesean decomposition of a function of $C(X)$ and the Dirichlearity of A, the mapping is defined if

(6) $$\operatorname{Re} f = 0 \Rightarrow \operatorname{Re} Vf = 0.$$

However, (6) is contained in

(7) $$\operatorname{Re} f \geqq 0 \Rightarrow \operatorname{Re} Vf \geqq 0,$$

which is nothing but the positivity of W.

5. A simple and elegant proof of (7) is established by Foiaş-Suciu [2]. Their proof is based on a fact pointed out by von Neumann [5; § 5.2 (23)]:

(8) $$\operatorname{Re} T \geqq 0 \Longleftrightarrow \|(T+1)\varphi\| \geqq \|(T-1)\varphi\|$$

for every $\varphi \in H$, which follows from

(9) $$4 \operatorname{Re} (T\varphi \mid \varphi) = \|(T+1)\varphi\|^2 - \|(T-1)\varphi\|^2.$$

If $\operatorname{Re} f \geqq 0$, then $f+1$ is invertible. If A is a Banach algebra, then $(f+1)^{-1} \in A$ by the Gelfand theory, and

(10) $$g = \frac{f-1}{f+1} \in A.$$

Since $\operatorname{Re} f \geqq 0$, $\|g\| \leqq 1$ and $f-1 = g(f+1)$. Since V satisfies (3),

$$\|(Vf-1)\varphi\| = \|Vg(Vf+1)\varphi\| \leqq \|(Vf+1)\varphi\|,$$

which proves (8) and so (7).

6. The original proof of Foiaş-Suciu [2] appealed the Naimark lattice dilation theorem which leaves some distance from (7); hence the above proof based on the Stinespring-Umegaki algebra dilation theorem is shorter and simpler.

Since the disk algebra is a Dirichlet algebra, the natural representation by a contraction T such as

(11) $$Vf=f(T)$$

is dilatable by Theorem 2, which is in turn the Sz.-Nagy strong dilation theorem, being taken as

(12) $$f_m(z)=z^m \qquad (m=0,1,2,\cdots).$$

If the algebra is not complete, then there is a slight trouble. In the usual way, A is completed and V is extended by the help of (3). However, sometimes, (10) is directly deducible, for example, if A is the algebra of all bounded rational functions.

Furthermore, by (8), (7) is deducible if V satisfies

(13) $$|f|\leqq|g| \Rightarrow \|Vf\varphi\|\leqq\|Vg\varphi\| \qquad (\varphi\in H).$$

The regular representation satisfies (13).

7. In the below, a few application of the theorem of Foiaş-Sucin will be discussed.

According to von Neumann [5], a (closed) set S of the complex numbers is a *spectral set* for an operator T if $\|f(T)\|\leqq 1$ for any rational function f with $\|f\|\leqq 1$ where the norm of f is the sup-norm on S.

If A is the algebra of all rational functions whose poles are not in S equipped with the sup-norm on S, then the definition is equivalent to state that (11) gives an operator representation of A on H. Hence the theorem of Foiaş-Suciu is applicable in the following theorem due to [4; III, Theorem 2]:

Theorem 3 (Lebow). *If A is a Dirichlet algebra of rational functions without poles in a spectral set X of T, then there exists a strong normal dilation N of T with $\sigma(N)\subset\partial X$, where $\sigma(N)$ is the spectrum of N and ∂X is the boundary of X.*

Being used f_1 in (12), if $N=Uf_1$, then N is normal since N lies in the homomorphic image of a commutative C^*-algebra, so that the first half of the theorem follows. Since a character on the image induces a character on $C(\partial X)$, $\sigma(N)$ is contained in the range of f_1 on ∂X; hence $\sigma(N)\subset\partial X$.

8. If a representation V of A is dilated in Theorem 2, then every functional ρ on $B(H)$ is transformed by U^*P^* on $C(X)$. Since P and U are contractive, $\|U^*P^*\rho\|\leqq\|\rho\|$. Especially, if $\rho=\varphi\otimes\psi$ for $\varphi,\psi\in H$ defined by

(14) $$\varphi\otimes\psi(S)=(S\varphi\mid\psi) \qquad (S\in B(H)),$$

then ρ corresponds to a dyad on H and $\|\varphi\otimes\psi\|=\|\varphi\|\|\psi\|$; hence $\mu(\varphi,\psi)=U^*P^*(\varphi\otimes\psi)$ is a regular Borel measure on X and satisfies

(15) $$\int_X f d\mu(\varphi,\psi)=(Vf\varphi\mid\psi)$$

and

(16) $$\|\mu(\varphi,\psi)\|\leqq\|\varphi\|\|\psi\|.$$

Conversely, if (15) and (16) are satisfied, then

$$\|Vf\varphi\|^2=\int_X f d\mu(\varphi, Vf\varphi)\leqq\|f\|\|\varphi\otimes Vf\varphi\|=\|f\|\|\varphi\|\|Vf\varphi\|,$$

so that $\|Vf\|\leqq\|f\|$ for every $f \in A$, which proves

Theorem 4. *If A is a Dirichlet algebra on X, and V is a homomorphism of A into $B(H)$ satisfying* (2). *Then V is an operator representation of A on H if and only if there is a regular Borel measure $\mu(\varphi, \psi)$ for every pair of φ and ψ in H which satisfies* (15) *and* (16).

Theorem 4 is a slight generalization of a theorem of Lebow [4; I, Theorem 1] which gives a necessary and sufficient condition for a spectral set of an operator.

9. The following well-known theorem is a main result in [5]:

Theorem 5 (von Neumann). *The unit disk is a spectral set for a contraction.*

In several occasions, cf. [8], the strong dilation theorem implies von Neumann's. However, the converse is also true under the light of Foiaş-Suciu's theorem.

If T is a contraction on H, then Theorems 2 and 5 imply that there is a normal strong dilation U of T with $\sigma(U)$ in the unit circle, so that U is a strong unitary dilation of T.

10. The *numerical range*

$$W(T)=\{(T\varphi\,|\,\varphi)\,;\,\|\varphi\|=1\} \tag{17}$$

of an operator T presents an opportunity of an another application of the general dilation theorem. Following after the naming of Fujii [3], an operator T is a *numeroid* if the closure $\bar{W}(T)$ of the numerical range of T is a spectral set for T. Then the dilation theorem implies there is a strong normal dilation N of T satisfying $\sigma(N)\subset\partial\bar{W}(T)$, so that

$$\bar{W}(T)\subset \bar{W}(N)=\operatorname{conv}\sigma(N)\subset\operatorname{conv}\partial\bar{W}(T)\subset \bar{W}(T)$$

by the convexity of $W(T)$, where $\operatorname{conv} S$ is the convex hull of S. Hence

$$\bar{W}(T)= \bar{W}(N). \tag{18}$$

This is the proof of the necessity part of the following theorem due to [6]:

Theorem 6 (Schreiber). *T is a numeroid if and only if there is a strong normal dilation N which satisfies* (18).

The following proof of the sufficiency is somewhat simpler than the original in [6]. Since the numerical range is convex and does not separate the plane, it is enough to prove that $\|q(T)\|\leqq\|q\|$ for every polynomial q by [4; p. 66]. Since N is a strong normal dilation of T, it is easy to check that $q(N)$ is a normal dilation of $q(T)$, so that $\|q(T)\|\leqq\|q(N)\|\leqq\|q\|$ by the spectral theorem.

11. If the algebra A of all rational functions with no poles in X is dense in $C(X)$, then X is called "*verdünnt*" in the sense of von Neumann [5; §§ 6. 4–6.5]. If X is "verdünnt" and V is an operator representation of a function algebra A on X, then V is directly extended to $C(X)$. By (7), if $f \geqq 0$, then

$$Vf = V \operatorname{Re} f = \operatorname{Re} Vf \geqq 0,$$

or V is positive on $C(X)$, so that V is a $*$-representation of $C(X)$ on H. Hence $VC(X)$ consists of normal operators. This shows

Theorem 7 (von Neumann). *If T has a "verdünnt" spectral set, then T is normal.*

The converse of the theorem is not true. It seems impossible that normal operators are characterized by purely spectral set terms.

12. Finally, an application of the theory of spectral sets on representations of C^*-algebras will be considered:

Theorem 8. *A C^*-algebra A is isometrically isomorphic to the algebra $B(E)$ of all operators on a Banach space E if and only if A is a factor of type I.*

If A is a factor of type I, then A is isometrically isomorphic with $B(H)$, so that the sufficiency is trivial. To prove the converse, it is remarked at first that the notion of spectral sets is not restricted on operators of Hilbert space. For any $a \in A$ with $\|a\| \leqq 1$, the unit disk is a spectral set for a by Theorem 5 being considered as an operator on a suitable Hilbert space; hence the hypothesis implies that the unit disk is a spectral set for any contractive operator on E. On the other hand, Foiaş [1; Theorem 2] established that E is a Hilbert space if each contractive operator on E has the unit disk as a spectral set. Hence Theorem 8 is proved.

Theorem 8 credits us, it is impossible that a factor of type II or III is represented isomorphically and isometrically by the algebra of all operators on a suitably chosen Banach space.

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