65. *Theorems on the Finite-dimensionality of Cohomology Groups. II*

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The purpose of this note is to present some theorems on the finite-dimensionality of cohomology groups attached to a system of linear differential equations, which supplement our preceding note [4]. The method of the proof is almost the same as that employed in the above quoted note, that is, it essentially relies on the local analysis of the solution sheaf of the system under consideration and the comparison of the topological structures which can be naturally introduced to the cohomology group under consideration in two ways. The local analysis of the solution sheaf may be used to prove that the two topologies coincide in the case which we treat in this note. Note that Professor Guillemin has announced results close to ours in a recent paper [2]. It seems that he has used the so-called sub-elliptic estimates, while our method essentially relies on Sato's theory of micro-functions. The employment of hyperfunctions, which allows us to employ linear (pseudo-)differential operators *of infinite order*, makes our results applicable to very general situations (, though we restrict ourselves to the real analytic category, that is, the manifold under consideration is real analytic and the coefficients of the differential operators are real analytic). As for the theory of micro-functions we refer to Sato [8] and Sato, Kawai and Kashiwara [9]. We also refer to Sato, Kawai and Kashiwara [9] for the theory of pseudo-differential operators of infinite order and the micro-local analysis of systems of linear (pseudo-) differential equations.

We use the same notations as in our previous note [4] and do not repeat their definitions. The details of this note will appear somewhere else with some technical improvements of the results.

The present writer expresses his hearty thanks to Mr. M. Kashiwara for many valuable discussions on the topics in the homological algebra especially the theory of derived category.

Theorem 1. *Let $\mathfrak{M}$ be an admissible system of linear differential equations defined on a compact manifold M.*

Case (i) *Assume that the generalized Levi form attached to V has q negative eigenvalues on $V_R = V \cap S^*M$. Then*

$$\dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{A}) = \dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{B}) < \infty \qquad \textit{for } i < q.$$

Hear $\mathcal{A}$ *and* $\mathcal{B}$ *denote the sheaf of germs of real analytic functions and that of hyperfunctions respectively.*

Case (ii) *Assume that there exists an integer q which is always equal to or larger than*

$$\sup_{(x,\eta)\in U} \dim_{(x,\eta)} \operatorname{proj} \mathfrak{M} - p,$$

where U is a suitable neighbourhood of $(x,\eta)\in V_R$ *and p is the number of positive eigenvalues of the generalized Levi form attached to V in U. Then*

$$\dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{A}) = \dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{B}) < \infty \qquad \text{for } i > q+1$$

and

$$\dim_C \operatorname{Ext}^{q+1}(M, \mathfrak{M}, \mathcal{B}) < \infty.$$

Theorem 2. *Let* $\mathfrak{M}$ *be an admissible elliptic system of linear differential equations defined on a compact manifold N. Let M be an open submanifold of N with real analytic boundary* ∂M.

Case (i) *Assume that the tangential system* $\mathfrak{N}$ *induced from* $\mathfrak{M}$ *on* ∂M *satisfies the conditions in Case* (i) *of Theorem* 1 (*when we regard* ∂M *as M in Theorem* 1). *Then*

$$\dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{A}) = \dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{B}) < \infty \qquad \text{for } i < q.$$

Case (ii) *Assume that the system* $\mathfrak{M}$ *satisfies the conditions in Case* (ii) *of Theorem* 1 (*when we regard* ∂M *as M in Theorem* 1). *Then*

$$\dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{A}) = \dim_C \operatorname{Ext}^i(M, \mathfrak{M}, \mathcal{B}) < \infty \qquad \text{for } i > q.$$

To prove this theorem we use the following cannonical isomorphism (1) which Kashiwara [3] has proved for general systems of linear differential equations by reducing the problem to the case of single linear differential equations treated in Komatsu and Kawai [6] by the aid of the theory of derived category. (Of course the isomorphism (1) holds only under a suitable non-characteristic condition, which is trivially satisfied in our case by the assumption of the ellipticity of $\mathfrak{M}$.)

(1) $$\boldsymbol{R}\,\mathcal{H}om_{'\mathcal{D}}(\mathfrak{N}, \mathcal{B}_{\partial M})[-1]\otimes\omega_{\partial M}\otimes\omega_N \rightleftarrows \boldsymbol{R}\underline{\Gamma}_{\partial M}\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathfrak{M}, \mathcal{B}_N),$$

where $'\mathcal{D}$ and $\mathcal{D}$ denote the sheaf of differential operators (of infinite order) on ∂M and N respectively, $\mathcal{B}_{\partial M}$ and $\mathcal{B}_N$ denote the sheaf of germs of hyperfunctions on ∂M and N respectively and $\omega_{\partial M}$ and ω_N denote the orientation bundle of ∂M and N respectively.

Theorem 3. *Let* $\mathfrak{M}$ *be an admissible elliptic system of linear differential equations with constant coefficients* (*defined on* $\boldsymbol{R}^n$). *Let M be a relatively compact open subset of* $\boldsymbol{R}^n$ *with real analytic boundary* ∂M. *Assume thath the tangential system* $\mathfrak{N}$ *induced from* $\mathfrak{M}$ *on* ∂M *satisfies the conditions in Case* (ii) *of Theorem* 1 *with* $q\geqq 0$ (*when we regard* ∂M *as M in Theorem* 1). *Then*

$$\dim_C H^i(M, \mathcal{S}) < \infty \qquad \text{for } i > q$$

where $\mathcal{S}$ *denotes the hyperfunction solution sheaf the system* $\mathfrak{M}$.

To prove this theorem we use the theorem of Komatsu [5] that

$$H^i(\Omega, \mathcal{S})=0 \qquad \text{for } i>0$$

as far as Ω is open and *convex*.

Remark. Note that the induced system $\mathfrak{N}$ is *not* with constant coefficients. The above employed method of reducing the problem to the tangential system may be regarded as a substitute of the celebrated "pie-nibbling method" due to Ehrenpreis [1], which reduces the problem to the formal adjoint system. (See also Malgrange [7].) Though this theorem is essentially only a very special case of Theorem 2, we have presented Theorem 3 independently of Theorem 2 in order to lay stress on the fact that it is better to investigate the system of general (pseudo-)differential equations even when one is concerned only with the system of linear differential equations *with constant coefficients*.

References

[1] Ehrenpreis, L.: Some applications of the theory of distributions to several complex variables. Seminar on Analytic Functions, **1**, 65–79, Princeton (1957).

[2] Guillemin, V.: On subelliptic estimates for complexes. Proc. Nice Congress, **2**, 227–230 (1971).

[3] Kashiwara, M.: Algebraic Study of the System of Partial Differential Equations. Master's thesis presented to Univ. of Tokyo (1971) (in Japanese).

[4] Kawai, T.: Theorems on the finite-dimensionality of cohomology groups. I. Proc. Japan Acad., **48**, 70–72 (1972).

[5] Komatsu, H.: Resolutions by hyperfunctions of sheaves of solutions of differential equations with constant coefficients. Math. Ann., **176**, 77–86 (1968).

[6] Komatsu, H., and T. Kawai: Boundary values of hyperfunction solutions of linear partial differential equations. Publ. RIMS, Kyoto Univ., **7**, 95–104 (1971).

[7] Malgrange, B.: Some remarks on the notion of convexity for differential operators. Differential Analysis, Tata Institute, 163–174 (1964).

[8] Sato, M.: Structure of hyperfunctions. Sûgaku no Ayumi, **15**, 9–72 (1970) (Notes by Kashiwara, in Japanese).

[9] Sato, M., T. Kawai, and M. Kashiwara: On pseudo-differential equations in hyperfunction theory (to appear in the report of Katata symposium).