88. *Abelian Projections over a von Neumann Subalgebra*

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1. In the theory of von Neumann algebras the notions of abelian and minimal projections play important roles.

In this note we shall introduce that a projection of a von Neumann algebra is minimal or abelian over a von Neumann subalgebra. These notions are generalizations of minimal and abelian projections, and they are same if the von Neumann subalgebra is included in the center. In this note, we shall prove that some elementary properties of minimal and abelian projections are preserved under our generalizations. Furthermore, we shall obtain certain conditions that the support projection of a normal state is minimal or abelian over certain von Neumann subalgebras.

We shall use the terminology due to Dixmier [2] throughout the note without further explanations.

2. In the sequel, let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$. Denote $\mathcal{B}^c$ the relative commutant $\mathcal{B}'\cap\mathcal{A}$ of $\mathcal{B}$ and by $\bar{E}$ the $\mathcal{B}$-support of a projection E in $\mathcal{A}$, that is, $\bar{E}$ is the infimum of projections in $\mathcal{B}$ which dominate E.

Lemma 1. *If $\bar{E}P$ is a projection for a projection P in $\mathcal{B}$ and a projection E in $\mathcal{B}^c$, then $\bar{E}P=\overline{EP}$.*

Proof. If $\bar{E}P\neq\overline{EP}$, then there exists a projection Q in $\mathcal{B}$ such that

$$\bar{E}P>Q\geqq EP.$$

Then $Q+\bar{E}(1-P)$ is a projection in $\mathcal{B}$ and

$$\bar{E}>Q+\bar{E}(1-P)\geqq E,$$

which is a contradiction.

Definition 1. A projection $E\in\mathcal{A}$ is called to be *abelian over* $\mathcal{B}$ if $E\in\mathcal{B}^c$ and, for any projection $P\in\mathcal{A}$ which is dominated by E, there exists a projection $Q\in\mathcal{B}$ such that $P=QE$.

Remark. If $\mathcal{B}$ is the center of $\mathcal{A}$, a projection abelian over $\mathcal{B}$ is abelian in the usual sense, cf. [2].

If $\mathcal{A}$ is an abelian von Neumann algebra, the notion that a projection is abelian over $\mathcal{B}$ is introduced by Dye [3].

Lemma 2. *If a projection $E\in\mathcal{A}$ is abelian over $\mathcal{B}$, then a projection $P\leqq E$ is written in $P=\bar{P}E$.*

Proof. If $P \leqq E$, then there is a projection $Q \in \mathcal{B}$ such that $P = QE$. Since $\bar{P} \leqq Q$, we have

$$P = \bar{P}P = \bar{P}QE = \bar{P}E.$$

Lemma 3. *If a projection E is abelian over $\mathcal{B}$ and F is a projection in $\mathcal{B}^c$ such that $F \prec E$ (mod. $\mathcal{B}^c$), then F is abelian over $\mathcal{B}$.*

Proof. For such projections E and F, there exists a partially isometric operator V in $\mathcal{B}^c$ such that

$$F = VV^* \quad \text{and} \quad V^*V \leqq E.$$

For a projection $P \in \mathcal{A}$ dominated by F, V^*PV is a projection in $\mathcal{A}$ dominated by V^*V. Since $\overline{V^*PV} = \bar{P}$ and V^*V is abelian over $\mathcal{B}$, we have $P = \bar{P}F$ by Lemma 2.

Proposition 1. *Let $\mathcal{B}$ by a normal von Neumann subalgebra of $\mathcal{A}$. If a projection $E \in \mathcal{A}$ is abelian over $\mathcal{B}$ and F is a projection in $\mathcal{B}^c$ such that $\bar{E} \leqq \bar{F}$, then $E \prec F$ (mod. $\mathcal{B}^c$).*

Proof. By the comparability theorem, there exists a projection G in the center of $\mathcal{B}^c$ such that

$$EG \succ FG \quad \text{and} \quad E(1-G) \prec F(1-G) \quad (\text{mod. } \mathcal{B}^c).$$

Clearly, G belongs to the center of $\mathcal{B}$. Let P be a projection in $\mathcal{B}^c$ such that $EG \geqq P \sim FG$ (mod. $\mathcal{B}^c$). Then Lemma 1 , Lemma 2 and the assumption imply

$$P = \bar{P}E = \bar{F}GE = GE.$$

Hence $E \prec F$ (mod. $\mathcal{B}^c$).

Corollary 1. *Let E and F be projections in $\mathcal{A}$ such that the central support of E is dominated by the central support of F. If E is abelian, then $E \prec F$ (mod. $\mathcal{A}$).*

Corollary 2. *Let $\mathcal{B}$ be a normal von Neumann subalgebra of $\mathcal{A}$. If two projections E and F in $\mathcal{A}$ are abelian over $\mathcal{B}$, then*

$$\bar{F} = \bar{E} \quad \textit{if and only if} \quad E \sim F \ (\text{mod. } \mathcal{B}^c).$$

Definition 2. A projection $E \in \mathcal{A}$ is called to be *minimal over* $\mathcal{B}$ if $E \in \mathcal{B}^c$ and there does not exist a projection $P \in \mathcal{A}$ such that $P \leqq E$, $\bar{E} = \bar{P}$ and $P \neq E$.

If a projection of $\mathcal{A}$ is abelian over $\mathcal{B}$, then it is minimal over $\mathcal{B}$.

Remark. If $\mathcal{B}$ is contained in the center of $\mathcal{A}$, then the definition is introduced by Guichardet [4].

Lemma 4. *If a projection $E \in \mathcal{A}$ is minimal over $\mathcal{B}$, then a projection $F \in \mathcal{B}^c$ with $F \prec E$ (mod. $\mathcal{B}^c$) is minimal over $\mathcal{B}$.*

Proof. It is clear that a projection $G \in \mathcal{B}^c$ with $G \sim E$ (mod. $\mathcal{B}^c$) is minimal over $\mathcal{B}$. Then there is no loss of generality in assuming that F is dominated by E. If F is not minimal over $\mathcal{B}$, then there exists a projection $Q \in \mathcal{A}$ such that $F \geqq Q$, $F \neq Q$ and $\bar{F} = \bar{Q}$. Put $E_1 = E - F$. Then

$$E = E_1 + F = \sup\{E_1, F\},$$

so that we have

$$\bar{E}=\sup\{\bar{E}_1,\bar{F}\}=\sup\{\bar{E}_1,\bar{Q}\}=\overline{E_1+Q}.$$

Since E is minimal over $\mathcal{B}$, $E=E_1+Q<E$, which is a contradiction.

Proposition 2 (Guichardet). *If $\mathcal{B}$ is contained in the center of $\mathcal{A}$, then a projection $E\in\mathcal{A}$ is minimal over $\mathcal{B}$ if and only if E is abelian over $\mathcal{B}$.*

Proof. It is sufficient to prove that if E is minimal over $\mathcal{B}$ then E is abelian over $\mathcal{B}$. Let P be a projection in $\mathcal{A}$ with $P\leqq E$. Since $\mathcal{B}$ is contained in the center of $\mathcal{A}$, $E\bar{P}$ is a projection in $\mathcal{B}^c$ satisfying $E\geqq E\bar{P}\geqq P$. By Lemma 4, $E\bar{P}$ is minimal over $\mathcal{B}$, and

$$\overline{E\bar{P}}=\bar{E}\bar{P}=\bar{P}$$

by Lemma 1, so we have $P=\bar{P}E$.

3. In this section, we shall give a simple application of the minimality over $\mathcal{B}$ of a projection in $\mathcal{A}$ to the support of a normal state on $\mathcal{A}$.

Let E_φ be the support of a normal state φ on $\mathcal{A}$. Since E_φ is the infimum of projections in $\mathcal{A}$ such that $\varphi(E_\varphi)=1$, we have

Proposition 3. *Let φ be a normal state on $\mathcal{A}$, then*

$$\overline{E_\varphi}=E_{\phi|\mathcal{B}},$$

where $\varphi|_{\mathcal{B}}$ is the restriction of φ to $\mathcal{B}$.

The following definition is due to Umegaki [5]:

Definition 3. A normal state φ on $\mathcal{A}$ satisfying

$$\varphi(AB)=\varphi(BA)$$

for every $A\in\mathcal{A}$ and $B\in\mathcal{B}$ is called a *$\mathcal{B}$-tracelet.*

Lemma 5. *If φ is a $\mathcal{B}$-tracelet, then E_φ belongs to $\mathcal{B}^c$.*

Proof. The left kernel

$$\mathcal{M}=\{T\in\mathcal{A};\ \varphi(T^*T)=0\}$$

is a σ-weakly closed left ideal in $\mathcal{A}$. If E is a projection in $\mathcal{A}$ with $\mathcal{M}=\mathcal{A}E$, cf. [2; p. 45], then for any projection $F\in\mathcal{B}$,

$$[E,F]=EF-FE\in\mathcal{M},$$

since φ is a $\mathcal{B}$-tracelet. Therefore, $E_\varphi=1-E$ commutes with $\mathcal{B}$, so that $E_\varphi\in\mathcal{B}^c$.

For any pair of normal states φ and ψ, we call φ being *absolutely continuous* with respect to ψ if $\psi(T)=0$ for $T\geqq 0$ implies $\varphi(T)=0$. If φ is absolutely continuous with respect to ψ, then we shall denote by $\varphi\prec\psi$. If $\varphi\prec\psi$ and $\psi\prec\varphi$ then we shall call that φ is *equivalent* to ψ and denote by $\varphi\sim\psi$.

Theorem 1. *Let φ be a $\mathcal{B}$-tracelet. E_φ is minimal over $\mathcal{B}$ if and only if φ satisfies the following condition:*

(*) $\qquad \varphi\succ\psi$ *and* $\varphi|_{\mathcal{B}}\sim\psi|_{\mathcal{B}}$ *imply* $\varphi\sim\psi$,

for any normal state ψ on $\mathcal{A}$.

Proof. By the definition,

$$\varphi\succ\psi \quad \text{if and only if} \quad E_\varphi\geqq E_\psi.$$

Proposition 3 implies that φ satisfies the condition (*) if E_φ is minimal over $\mathcal{B}$.

Conversely, assume that the condition (*) is satisfied. Take a projection $P \in \mathcal{A}$ such that

$$P \leqq E_\varphi \quad \text{and} \quad \bar{P} = \bar{E}_\varphi,$$

and define a normal state ψ by

$$\psi(T) = \frac{\varphi(PTP)}{\varphi(P)},$$

then $E_\psi = P$. Therefore, we have

$$\varphi \succ \psi \quad \text{and} \quad \varphi|_{\mathcal{B}} \sim \psi|_{\mathcal{B}}.$$

Hence we have $\varphi \sim \psi$, that is, $E_\varphi = E_\psi = P$.

Corollary 1. *The following conditions are equivalent:*

(i) *E_φ is a minimal projection in $\mathcal{A}$,*

(ii) *for any normal state ψ on $\mathcal{A}$, $\varphi \succ \psi$ implies $\varphi \sim \psi$,*

(iii) *φ is an extreme point of the set of all normal states of $\mathcal{A}$.*

Proof. In Theorem 1, let $\mathcal{B}$ be the scalar multiples of the identity operator, then we have the equivalence of the conditions (i) and (ii). The equivalence of the conditions (i) and (iii) are proved in [1; Proposition 2.1].

An automorphism g of $\mathcal{A}$ is called to be *quasiinvariant* under a normal state φ if $\varphi \succ \varphi_g$ where $\varphi_g(T) = \varphi(T^g)$ for every $T \in \mathcal{A}$.

Corollary 2. *Let φ be a normalized normal trace on $\mathcal{A}$, g an quasiinvariant automorphism of $\mathcal{A}$ under φ and $\mathcal{Z}$ a fixed algebra of g, that is,*

$$\mathcal{Z} = \{T \in \mathcal{A}\,;\, T^g = T\}.$$

If E_φ is minimal over $\mathcal{Z}$, then $\varphi \sim \varphi_g$.

Lemma 6. *Let φ be a $\mathcal{B}$-tracelet. If E_φ is abelian over $\mathcal{B}$, then φ satisfies the following condition: For normal state ψ on $\mathcal{A}$,*

(**) $$\varphi \succ \psi,\ \varphi|_{\mathcal{B}} = \psi|_{\mathcal{B}} \quad \textit{imply} \quad \varphi = \psi.$$

Proof. If E_φ is abelian over $\mathcal{B}$, then E_φ is minimal over $\mathcal{B}$. Hence φ satisfies the condition (*) in Theorem 1. Let ψ be a normal state such that

$$\varphi \succ \psi \quad \text{and} \quad \varphi|_{\mathcal{B}} = \psi|_{\mathcal{B}},$$

then by Theorem 1 we have $E_\varphi = E_\psi$.

On the other hand, if E_φ is abelian over $\mathcal{B}$, then we have

$$E_\varphi \mathcal{A} E_\varphi = E_\varphi \mathcal{B} E_\varphi.$$

Hence, by an immediate calculation, Lemma 6 is proved.

From Lemma 6, we have immediately the following theorem:

Theorem 2. *Let φ be a $\mathcal{B}$-tracelet. If E_φ is abelian over $\mathcal{B}$, then φ is an extreme point of the set of all normal states whose restrictions on $\mathcal{B}$ coincide with the restriction $\varphi|_{\mathcal{B}}$ of φ on $\mathcal{B}$.*

Let $\mathcal{B}$ be the scalar multiples of the identity operator in Theorem

2, then the implication (i)$\Rightarrow$(ii) in Corollary 1 of Theorem 1 follows immediately without appealing [1; Proposition 2.1].

References

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