84. *Rigidity for Elliptic Isometric Imbeddings*

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Introduction. We say that an isometric imbedding f of a Riemannian manifold M to the Euclidean space $\boldsymbol{R}^m$ is elliptic if the imbedding f is generic in a suitable sense and if the differential operator L associated with f is elliptic. We then establish a rigidity theorem (Theorem 2) for elliptic isometric imbeddings of compact Riemannian manifolds to $\boldsymbol{R}^m$. Furthermore we apply this result to the rigidity problem associated with the canonical isometric imbedding f of a compact hermitian symmetric space $M=G/H$ to the Euclidean space $\boldsymbol{R}^m$, where $m=\dim G$ (see Theorems 3 and 4). Theorem 4 partially generalizes the classical theorem of Cohn-Vossen.

1. Throughout the present paper we shall always assume the differentiability of class C^∞.

Let M be an n-dimensional manifold. T denotes the tangent bundle of M and T^* its dual. S^2T^* denotes the vector bundle of symmetric tensors of type $\binom{0}{2}$ on M. Given a vector bundle E on M, $\Gamma(E)$ denotes the space of cross-sections of E. Let $\boldsymbol{R}^m$ be the m-dimensional Euclidean space. $\langle\ ,\ \rangle$ denotes the inner product on $\boldsymbol{R}^m$ as a Euclidean vector space.

Let $\Gamma(M, m)$ be the vector space of all the maps $\boldsymbol{u}$ of M to $\boldsymbol{R}^m$ and $\mathfrak{E}$ the subset of $\Gamma(M, m)$ consisting of all the imbeddings f of M to $\boldsymbol{R}^m$. We assume $\mathfrak{E}\neq\phi$. For any $f\in\mathfrak{E}$, we denote by $\Phi(f)$ the Riemannian metric on M induced by the imbedding f:

$$\Phi(f)=\langle df, df\rangle=\sum_k (df_k)^2,$$

where $f=(f_1, \cdots, f_m)$. Then the assignment $f\to\Phi(f)$ gives a map Φ of the set $\mathfrak{E}$ to the set $\mathfrak{M}$ of all the Riemannian metrics on M. For any $f\in\mathfrak{E}$, we define a differential operator Φ_{*f} of $\Gamma(M, m)$ to $\Gamma(S^2T^*)$, the differential of the map Φ at f, by

$$\Phi_{*f}(\boldsymbol{u})=2\langle df, d\boldsymbol{u}\rangle \quad (\boldsymbol{u}\in\Gamma(M, m)).$$

2. Let f be an imbedding of M to $\boldsymbol{R}^m$. We put $\nu=\Phi(f)$. Let N be the normal vector bundle on M associated with the imbedding f; the fibre N_p over a point p of M may be identified with a subspace of the Euclidean vector space $\boldsymbol{R}^m$. It is well known that, for any vectors $X, Y\in T_p$, the derivative $\nabla_X\nabla_Y f$ is in the subspace N_p of $\boldsymbol{R}^m$, where ∇ is the covariant differentiation associated with the Riemannian metric ν.

Let us now consider the following condition (C) for the imbedding f: At each point $p \in M$, the fibre N_p of the normal bundle N is generated by the vectors of the form $\nabla_X \nabla_Y f (X, Y \in T_p)$.

For any $\boldsymbol{a} \in N_p$, define an element $\theta_{\boldsymbol{a}}$ of S^2T^* by

$$\theta_{\boldsymbol{a}}(X, Y) = \langle \boldsymbol{a}, \nabla_X \nabla_Y \boldsymbol{f} \rangle,$$

which is usually called the second fundamental form of f in the direction $\boldsymbol{a}$. Then we see that f satisfies condition (C) if and only if the map $\theta : N \ni \boldsymbol{a} \to \theta_{\boldsymbol{a}} \in S^2T^*$ is injective.

3. In what follows, we assume that the imbedding f satisfies condition (C). By the above remark, the image of the map θ forms a subbundle $\tilde{N}$ of S^2T^*, which is called the bundle of second fundamental forms of f.

We define a differential operator D of $\Gamma(T^*)$ to $\Gamma(S^2T^*)$ by

$$(D\varphi)(X, Y) = (\nabla_X \varphi)(Y) + (\nabla_Y \varphi)(X) \qquad (\varphi \in \Gamma(T^*))$$

and denote by π the projection of S^2T^* onto the factor bundle $S^2T^*/\tilde{N}$. Then the composition $L = \pi \circ D$ is a differential operator of $\Gamma(T^*)$ to $\Gamma(S^2T^*/\tilde{N})$.

Theorem 1. *Let f be an imbedding of M to $\boldsymbol{R}^m$ which satisfies condition* (C), *and let α be any element of $\Gamma(S^2T^*)$. Then the solutions $\boldsymbol{u}$ of the equation $\Phi_{*f}(\boldsymbol{u}) = \alpha$ are in a one-to-one correspondence with the solutions φ of the equation $L\varphi = \pi\alpha$. Moreover the correspondence $\boldsymbol{u} \to \varphi$ is given by the relation $\varphi = \langle \boldsymbol{u}, d\boldsymbol{f} \rangle$.*

4. Let f be as in Theorem 1. For any covector $\xi \in T_p^*$, the symbol $\sigma(D)_\xi : T_p^* \to S^2T_p^*$ at ξ of the operator D is given by

$$\sigma(D)_\xi \eta = 2\xi \cdot \eta \qquad (\eta \in T_p^*),$$

where $\xi \cdot \eta$ is the symmetric product of ξ and η. It follows that the symbol $\sigma(L)_\xi : T_p^* \to S^2T_p^*/\tilde{N}_p$ at ξ of the operator L is given by

$$\sigma(L)_\xi \eta = 2\pi(\xi \cdot \eta).$$

We say that a subbundle of S^2T^* is elliptic if it contains no non-zero elements of the form $\xi \cdot \eta (\xi \cdot \eta \in T_p^*)$. Then we have the following

Proposition. *The differential operator L associated with the imbedding f is elliptic if and only if the bundle $\tilde{N}$ of second fundamental forms of f is elliptic.*

An imbedding f of M to $\boldsymbol{R}^m$ will be called elliptic if it satisfies condition (C) and if the bundle $\tilde{N}$ is elliptic.

5. Let us now introduce the C^3-topology in the set $\mathfrak{E}$ of imbeddings and denote by $\mathfrak{E}_{C^3}$ the topological space thus obtained. For any $f \in \mathfrak{E}$, let $\rho(f)$ denote the dimension of the solution space of the equation $L\varphi = 0$. Then we have

$$\rho(f) \geqq \frac{1}{2} m(m+1),$$

the dimension of the Euclidean transformation group $E(m)$ of $\boldsymbol{R}^m$.

Theorem 2. *Let f be an imbedding of M to $\boldsymbol{R}^m$. We assume that M is compact, f is elliptic and that the dimension $\rho(f)$ is just equal to $\frac{1}{2}m(m+1)$. Then there exists a neighborhood $U(f)$ of f in $\mathfrak{E}_{C^3}$ having the following property: If $f', f'' \in U(f)$ and if $\Phi(f')=\Phi(f'')$, there exists a unique Euclidean transformation a of $\boldsymbol{R}^m$ such that $f''=af'$.*

Proof of Theorem 2 uses Theorem 1 and the theory of elliptic differential operators.

6. Let $M=G/H$ be a hermitian symmetric homogeneous space, where G is connected. We assume that M is of compact type.

Let $\mathfrak{g}$ be the Lie algebra of G and B its Killing form. Note that B is negative definite. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B. As usual $\mathfrak{m}$ may be identified with the tangent space T_o to M at the origin o of the homogeneous space. This being said, there is a unique G-invariant Riemannian metric ν on M such that $\nu(X, Y)=-B(X, Y)$ $(X, Y \in T_o)$. Let I be the almost complex structure of the complex manifold M. Then it is well known that there is a unique element Z_0 of the centre of $\mathfrak{h}$ such that $H=\{a \in G \mid \mathrm{ad}\, a\, Z_0=Z_0\}$ and such that $IX=[Z_0, X]$ $(X \in T_o)$.

Let us now define an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by $\langle X, Y\rangle=-B(X, Y)$ $(X, Y \in \mathfrak{g})$. Thus $\mathfrak{g}$ may be considered as the Euclidean space $\boldsymbol{R}^m$, where $m=\dim \mathfrak{g}$. The map $G \ni a \rightarrow \mathrm{ad}\, a\, Z_0 \in \mathfrak{g}$ induces a map f of M to $\mathfrak{g}$, and it can be shown that f is an isometric imbedding of the hermitian symmetric space (M, ν) to the Euclidean space $\mathfrak{g}=\boldsymbol{R}^m$ (cf. Lichnerowicz, Géométrie des groupes de Transformations, Dunod, Paris, 1958).

Theorem 3. *The isometric imbedding $f:(M, \nu) \rightarrow \mathfrak{g}$ satisfies condition* (C) *and is elliptic, and $\rho(f)$ is equal to $\frac{1}{2}m(m+1)$.*

Proof of Theorem 3 needs Theorem 1, Proposition and some calculuses on the Laplacian Δ of the Kählerian manifold M.

By Theorems 2 and 3 we have

Theorem 4. *Let f be the isometric imbedding of the compact hermitian symmetric space (M, ν) to the Euclidean space $\mathfrak{g}=\boldsymbol{R}^m$. Then there exists a neighborhood $U(f)$ of f in $\mathfrak{E}_{C^3}$ having the following property: If $f', f'' \in U(f)$ and if $\Phi(f')=\Phi(f'')$, there exists a unique Euclidean transformation a of $\mathfrak{g}$ such that $f''=af'$.*

The detailed results together with their complete proofs will be published elsewhere.