134. On a Fine Capacity Related to a Symmetric Markov Process

By Masaru TAKANO
Tokyo University of Education

(Comm. by Kôsaku YOSIDA, M. J. A., Oct. 12, 1972)

§1. Introduction and main results.

Let X be a locally compact Hausdorff space with a countable base and m be a positive Radon measure on X. Let $M=(X_t, P_x, \zeta)$ be an m-symmetric standard process on X. Throughout this paper we make the following assumption:

(A) The measure m is a reference measure for M.

By virtue of (A) and the symmetry of M, it follows from Theorem 1.4 in [1; Chap. 6] that M is self-dual in the sense of [1; Chap. 6]. Further polarity and semipolarity of a set are equivalent (Proposition 4.10 in [1; Chap. 6]). Hence every fine Borel set is nearly Borel because under (A) every fine Borel set is the union of a Borel set and a semipolar set ([1; Chap. 5]).

The expression "*q.e.*" will mean "except on a polar set". A function u defined q.e. on X is called *q.e. finely continuous* if there exists a nearly Borel polar set B such that u is finely continuous on $X-B$. Denote by $(X, m, \mathcal{F}, \mathcal{E})$ the Dirichlet space generated by the m-symmetric resolvent $\{G_\alpha, \alpha>0\}$ of M in the sense of Fukushima [2; §2]. Our main results are the following.

Theorem 1. *Every function in $\mathcal{F}$ has a q.e. finely continuous modification: for every $u \in \mathcal{F}$, there exists a q.e. finely continuous function u^* such that $u^*=u$ m-a.e.*

Denote by $\mathcal{F}^*$ the set of all q.e. finely continuous modifications of functions of $\mathcal{F}$. For each $\alpha>0$, set $\mathcal{E}_\alpha(u, v)=\mathcal{E}(u, v)+\alpha(u, v)$ for $u, v \in \mathcal{F}$, where (u, v) denotes the inner product in $L^2=L^2(X, m)$.

Theorem 2. *If $\{u_n\}$ is a Cauchy sequence in the Hilbert space $(\mathcal{F}^*, \mathcal{E}_1)$, then there exists a subsequence which converges q.e. on X to a function $u \in \mathcal{F}^*$. Furthermore $\{u_n\}$ converges to u with $\mathcal{E}_1$-norm.*

For a finely open set A, let

(1.1) $$\mathcal{L}_A=\{u \in \mathcal{F};\ u \geqq 1 \quad m\text{-a.e. on } A\}$$

and define

(1.2) $$\operatorname{cap}(A)=\inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u, u) \qquad \text{if } \mathcal{L}_A \neq \phi$$
$$=\infty \qquad \text{if } \mathcal{L}_A=\phi.$$

For any subset B of X, define

$$\text{(1.3)} \qquad \operatorname{cap}(B)=\inf_{B\subset A,\,A:\text{finely open}} \operatorname{cap}(A).$$

We call $\operatorname{cap}(B)$ the *fine capacity* of B. The fine capacity is a non-negative, countably subadditive Choquet capacity with respect to the fine topology. This can be verified in the same manner as in [3; § 2]. It is clear that any set of zero fine capacity is m-negligible. Furthermore we can assert the following.

Theorem 3. *A set N is of zero fine capacity if and only if N is polar.*

If a regular Dirichlet space is firstly given, then Fukushima [3] shows that it is generated by a certain Hunt process, establishing further several theorems which are analogous to the present ones. There the capacity is defined by means of open sets of the underlying topology rather than the fine topology.

We instead start with a general standard process. We do not know whether generally the Dirichlet space generated by the given standard process is regular or not. However our results show that, without any assumption of regularity, the results in [3] still hold if one passes from the usual capacity to the fine capacity.

§ 2. Proofs. Let $\mathcal{O}$ be the class of finely open sets A such that $\mathcal{L}_A\neq\phi$. For every $A\in\mathcal{O}$, there exists a unique element $p_A\in\mathcal{L}_A$ minimizing the quadratic form $\mathcal{E}_1(u,u)$ in $\mathcal{L}_A$, which satisfies that $\operatorname{cap}(A)=\mathcal{E}_1(p_A,p_A)$, $0\leqq p_A\leqq 1$ m-a.e. on X, $p_A=1$ m-a.e. on A and $\mathcal{E}_1(p_A,v)\geqq 0$ for every $v\in\mathcal{F}$ such as $v\geqq 0$ m-a.e. on A (cf. [3; § 1]).

Lemma 1. *If $A\in\mathcal{O}$ there exists a 1-excessive (consequently, finely continuous) function $\tilde{p}_A$ such that $\tilde{p}_A=p_A$ m-a.e. Further we have*

$$\text{(2.1)} \qquad \tilde{p}_A(x)=1 \qquad \text{for all } x\in A,$$

and

$$\text{(2.2)} \qquad \tilde{p}_A(x)\geqq E_x(e^{-\sigma_A}) \qquad \text{for all } x\in X.$$

Here σ_A is the first hitting time of A.

Proof. If $v\in L^2$ and $v\geqq 0$ m-a.e., then

$$\begin{aligned}(p_A-\alpha G_{\alpha+1}p_A, v)&=\mathcal{E}_{\alpha+1}(p_A, G_{\alpha+1}v)-\alpha(p_A, G_{\alpha+1}v)\\&=\mathcal{E}_1(p_A, G_{\alpha+1}v)\geqq 0\end{aligned}$$

because of the above mentioned property of p_A. Hence $p_A\geqq\alpha G_{\alpha+1}p_A$ m-a.e. and the everywhere defined function $\alpha G_{\alpha+1}p_A$ is increasing as $\alpha\to\infty$. It has the limit $\tilde{p}_A$ which is 1-excessive and equal to p_A m-a.e. The equality (2.1) follows from (A). The inequality (2.2) is a consequence of the equality (2.1) and Proposition 2.8 in [1; Chap. 2].

Lemma 2. *Any set of zero fine capacity is polar.*

Proof. If N is a set of zero fine capacity, there exists a decreasing sequence $\{A_n\}$ in $\mathcal{O}$ satisfying $A_n\supset N$ and $\operatorname{cap}(A_n)\downarrow 0$. Since $\operatorname{cap}(A_n)=\mathcal{E}_1(\tilde{p}_{A_n},\tilde{p}_{A_n})\geqq(\tilde{p}_{A_n},\tilde{p}_{A_n})$, $\liminf_{n\to\infty}\tilde{p}_{A_n}=0$ m-a.e. By the inequality (2.2),

$$0=E_x(\exp(-\lim_{n\to\infty}\sigma_{A_n}))\geqq E_x(\exp(-\sigma_{\cap A_n})) \qquad m\text{-a.e.}$$

Therefore, by virture of (A), we see that $\bigcap_{n=1}^{\infty}A_n$ is polar and so is N.

Owing to the above proof, the condition (A) and Proposition 3.2 in [1; Chap. 6], we see further that if $A_n\in\mathcal{O}$ and $\operatorname{cap}(A_n)\downarrow 0$, then

$$E_x(\exp(-\lim_{n\to\infty}\sigma_{A_n}))=0 \tag{2.3}$$

except on a polar set.

A function u defined except on a set of zero fine capacity is called *finely quasi-continuous* if for any $\varepsilon>0$ there exists a finely open set A such that $\operatorname{cap}(A)<\varepsilon$ and the restriction of u to $X-A$ is finely continuous. Let $\boldsymbol{B}$ be the set of all bounded Borel measurable functions. Every function in $G_1(L^2\cap\boldsymbol{B})$ then is finely continuous and $G_1(L^2\cap\boldsymbol{B})$ is dense in the Hilbert space $(\mathcal{F},\mathcal{E}_1)$. Therefore, taking $G_1(L^2\cap\boldsymbol{B})$ in place of $C(X)$ in [3; Theorem 1.3], we see that if u belongs to $\mathcal{F}$, then there exists a finely quasi-continuous function u' satisfying $u=u'$ m-a.e.

Lemma 3. *A finely quasi-continuous function is q.e. finely continuous.*

Proof. If u is a finely quasi-continuous function, there exists a decreasing sequence $\{A_n\}$ in $\mathcal{O}$ such that $\operatorname{cap}(A_n)\downarrow 0$ and restriction of u to $X-A_n$ is finely continuous on $X-A_n$. In view of the proof of Lemma 2 and the fact (2.3), $\bigcap_{n=1}^{\infty}A_n$ is a nearly Borel polar set and $P_x(\lim_{n\to\infty}\sigma_{A_n}=\infty)=1$ except on a polar set. Let C be a nearly Borel polar set containing the exceptional polar set. The set $B=\bigcap_{n=1}^{\infty}(C\cup A_n)$ is again a nearly Borel polar set.

Fix any point x of $X-B$. Since $P_y(\lim_{n\to\infty}\sigma_{C\cup A_n}=\infty)=1$ for every y of $X-B$, $P_x(\sigma_{C\cup A_n}>0)=1$ for sufficiently large n by Blumenthal's 0-1 law. Hence for sufficiently large n, $X-(C\cup A_n)$ is a fine neighborhood of x, which implies that u is finely continuous at x.

Proof of Theorems 1 and 2. Theorem 1 is a consequence of Lemma 3, Theorem 2 follows from Lemma 2 and Lemma 1.2 in [3].

Now we turn to the proof of Theorem 3. For a fixed finely open set $A\subset X$, the resolvent operator $G_\alpha^A u(x)\equiv E_x\left(\int_0^{\sigma_A}e^{-\alpha t}u(X_t)dt\right)$ also generates a Dirichlet space $(X,m,\mathcal{F}^{(A)},\mathcal{E}^{(A)})$.

Lemma 4. *The space $\mathcal{F}^{(A)}$ is a subset of $\mathcal{F}$ and $\mathcal{E}^{(A)}$ is the restriction of $\mathcal{E}$ to $\mathcal{F}^{(A)}$:*

$$\mathcal{E}^{(A)}(u,v)=\mathcal{E}(u,v) \qquad \text{for } u,v\in\mathcal{F}^{(A)}. \tag{2.4}$$

This lemma is proved in [3; § 4] if A is open or closed in the underlying topology. It is easy to see that it is true even if A is finely open. By means of this lemma, $\mathcal{F}^{(A)}$ is a closed subspace of the Hilbert space $(\mathcal{F},\mathcal{E}_1)$. Denote by $\mathcal{H}^A$ the orthogonal complement of $\mathcal{F}^{(A)}$ in $(\mathcal{F},\mathcal{E}_1)$. Let $(\mathcal{H}^A)^*$ be the set of all q.e. finely continuous modifications of functions of $\mathcal{H}^A$. Denote by $\boldsymbol{B}^n$ the set of all bounded nearly Borel measurable functions. We define

(2.5) $$H_\alpha^A u(x) = E_x(e^{-\alpha\sigma_A} u(X_{\sigma_A})) \qquad \text{for } u \in \boldsymbol{B}^n.$$

Lemma 5. *If $u \in \mathcal{F}^* \cap \boldsymbol{B}^n$, then $H_1^A u$ is the projection of u on the space $(\mathcal{H}^A)^*$.*

Proof (see [3; Lemma 3.4]). We only note the following: By Theorem 2, a suitable subsequence of $\{\alpha G_\alpha u\}$ converges to some $u' \in \mathcal{F}^*$ q.e. and $u' = u$ m-a.e. We have $u' = u$ q.e. by virture of (A). Therefore the subsequence converges to u q.e.

As consequences of Lemma 5, we see the following facts.

1° For every $A \in \mathcal{O}$, $e_A(x) \equiv E_x(e^{-\sigma_A}) (= E_x(e^{-\sigma_A}\tilde{p}_A(X_{\sigma_A})))$ belongs to $(\mathcal{H}^A)^*$ and hence to $\mathcal{F}^*$.

2° If $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{O}$, then $\{e_{A_n}\}$ is a Cauchy sequence with $\mathcal{E}_1$-norm.

The next lemma combined with Lemma 2 completes the proof of Theorem 3.

Lemma 6. *A polar set N is of zero fine capacity.*

Proof. We may assume that N is nearly Borel. Moreover, if $\bar{N}$ is compact, there exists a bounded continuous function $f \in L^2$ which is larger than 1 on $\bar{N}$. Then $N \subset \bigcup_{n=1}^\infty \{x; nG_n f(x) > 1\}$ and each set $\{x; nG_n f(x) > 1\}$ belongs to $\mathcal{O}$. Since the space X is σ-compact, it suffices to consider the case that N is a nearly Borel polar set whose fine capacity is finite. Let h be a strictly positive function in $L^1(X, m)$ and set $h \cdot m(E) = \int_E h(x) m(dx)$. Then there exists a decreasing sequence $\{A_n\}$ of sets in $\mathcal{O}$ such that $A_n \supset N$ and $\sigma_{A_n} \uparrow \sigma_N$ $P_{h\cdot m}$-a.s., consequently $\sigma_{A_n} \uparrow \infty$ $P_{h\cdot m}$-a.s. Therefore,

$$\int \lim_{n\to\infty} e_{A_n}(x) h \cdot m(dx) = \lim_{n\to\infty} E_{h\cdot m}(\exp(-\sigma_{A_n})) = 0.$$

Hence $\lim_{n\to\infty} e_{A_n}(x) = 0$ m-a.e. Then by the above fact 2° and Theorem 2, $\{e_{A_n}\}$ converges to 0 with $\mathcal{E}_1$-norm. Since $e_{A_n} = 1$ on A_n, we see that

$$\mathrm{cap}\,(N) \leqq \lim_{n\to\infty} \mathrm{cap}\,(A_n) \leqq \lim_{n\to\infty} \mathcal{E}_1(e_{A_n}, e_{A_n}) = 0.$$

References

[1] R. M. Blumenthal and R. K. Getoor: Markov Process and Potential Theory. Academic Press, New York (1968).

[2] M. Fukushima: Regular representation of Dirichlet space. Trans. Amer. Math. Soc., **155**, 455–473 (1971).

[3] ——: Dirichlet spaces and strong Markov processes. Trans. Amer. Math. Soc., **162**, 185–224 (1971).