8. *On Measurable Functions. II*

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In this part of the paper, some relations between the sets $\mathcal{H}$ and $\mathcal{G}$ stated in the introduction in Part I will be discussed.

3. The set of all measurable functions. Assumption 3.1. *M is a non-empty set and $\mathcal{S}$ is a ring of subsets of M.*

For a topological additive group K, throughout this section we shall use the following notations:

1) Let $G=J=\{0\}$ be the topological additive group consisting of only one element and define the product of $0\in G$ and $k\in K$ by $0\cdot k=0\in J$. Then the system (M, G, K, J) becomes an integral system and this integral system is denoted by $\Lambda(K)$.[1]

2) $\mathcal{F}(K)$ is the total functional group of $\Lambda(K)$.

Then $(\mathcal{S}, \mathcal{F}(K), J)$ is an abstract integral structure.

3) $\mathcal{G}(K)$ is the integral closure of K in $\mathcal{F}(K)$.

4) $\mathcal{G}_0(K)$ is the subgroup of $\mathcal{F}(K)$ generated by $\mathcal{S}K$.

Then $\mathcal{G}(K)$ is the $\mathcal{F}(K)$-completion of the closure of $\mathcal{G}_0(K)$ in $\mathcal{F}(K)$.

5) $\mathcal{V}(K)$ is the system of neighbourhoods of $0\in K$ and $\tilde{V}=\{f\mid f\in\mathcal{F}(K), f(M)\subset V\}$ for each $V\in\mathcal{V}(K)$.

Then $\{\tilde{V}\mid V\in\mathcal{V}(K)\}$ is a base of the system of neighbourhoods of $0\in\mathcal{F}(K)$.

Now we can state a property of $\mathcal{G}(K)$ corresponding to Theorem 2.1 in [1].

Theorem 3.1. *Let K_i, $i=1,2,\cdots,n$, be topological additive groups. Let D be a subspace of the product space $\prod_{i=1}^{n}K_i$ and φ a uniformly continuous map of D into a topological additive group K. Then, for $f_i\in\mathcal{G}(K_i)$, $i=1,2,\cdots,n$, such that $(f_1(x),\cdots,f_n(x))\in D$ for each $x\in M$, and for the map f of M into K defined by $f(x)=\varphi(f_1(x),\cdots,f_n(x))$ for each $x\in M$, it holds that $f\in\mathcal{G}(K)$.*

Proof. Let X be an element of $\mathcal{S}$. It suffices to show that $Xf\in\overline{\mathcal{G}_0(K)}$ or equivalently that $(Xf+\tilde{V})\cap\mathcal{G}_0(K)\neq\phi$ for any $V\in\mathcal{V}(K)$. The uniform continuity of φ implies the existence of $V_i\in\mathcal{V}(K_i)$, $i=1,2,\cdots,n$, satisfying the condition: $\varphi(x_1,\cdots,x_n)-\varphi(y_1,\cdots,y_n)\in V$

1) The topological additive groups G and J play no essential role here. These groups are introduced only for the sake of the definitions of $\mathcal{F}(K)$, $\mathcal{G}(K)$, etc. Therefore, G and J may be replaced by any other groups such that (M, G, K, J) becomes an integral system.

for elements $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ of D such that $x_i - y_i \in V_i$, $i=1,2,\cdots,n$. We have $U_i \in \mathcal{V}(K_i)$ such that $U_i - U_i \subset V_i$ for each i. Since $f_i \in \mathcal{G}(K_i)$, we have $g_i \in (Xf_i + \tilde{U}_i) \cap \mathcal{G}_0(K_i)$ and we can write $g_i = \sum_{j=1}^{m_i} X'_{ij} a_{ij}$ for some $a_{ij} \in K_i$ and $X'_{ij} \in \mathcal{S}$ such that $X'_{ij} X'_{ij'} = 0$ $(j \neq j')$. Put $X_{ij} = XX'_{ij}$ for $j \geqq 1$ and $X_{i0} = X + \sum_{j=1}^{m_i} X_{ij}$. Then, for each i, putting $a_{i0} = 0 \in K_i$ we have 1) $a_{ij} \in K_i$ and $X_{ij} \in \mathcal{S}$ for each $j=0,1,\cdots,m_i$, 2) $X_{ij} X_{ij'} = 0$ $(j \neq j')$, 3) $X = \sum_{j=0}^{m_i} X_{ij}$, and 4) $Xg_i = X \sum_{j=1}^{m_i} X'_{ij} a_{ij} = \sum_{j=1}^{m_i} X_{ij} a_{ij} = \sum_{j=0}^{m_i} X_{ij} a_{ij}$. Put $\Theta = \{\theta \mid \theta = (j_1, \cdots, j_n),\ 0 \leqq j_i \leqq m_i$ for each i, $X_{1j_1} X_{2j_2} \cdots X_{nj_n} \neq 0\}$ and for each $\theta = (j_1, \cdots, j_n) \in \Theta$ put $X_\theta = X_{1j_1} X_{2j_2} \cdots X_{nj_n}$. Let x_θ be a fixed element of X_θ for each $\theta \in \Theta$. Then putting $g = \sum_{\theta \in \Theta} X_\theta f(x_\theta)$ we have an element g of $\mathcal{G}_0(K)$.

Now it is sufficient to prove that $g \in Xf + \tilde{V}$ or that $g(x) - (Xf)(x) \in V$ for each $x \in M$. Since $g(x) = (Xf)(x) = 0$ for $x \notin X$, we may assume that $x \in X$. For each i, $x \in X = \sum_{j=0}^{m_i} X_{ij}$ implies the existence of k_i $(0 \leqq k_i \leqq m_i)$ such that $x \in X_{ik_i}$. Put $\lambda = (k_1, \cdots, k_n)$. Then $x \in X_{1k_1} \cdots X_{nk_n}$ implies $\lambda \in \Theta$. Since $(j_1, \cdots, j_n) \neq (j'_1, \cdots, j'_n)$ implies $(X_{1j_1} \cdots X_{nj_n})(X_{1j'_1} \cdots X_{nj'_n}) = 0$, it follows that $g(x_\lambda) = f(x_\lambda)$. The relations $g_i \in Xf_i + \tilde{U}_i$ and $x, x_\lambda \in X$ imply that $g_i(x) - f_i(x) = g_i(x) - (Xf_i)(x) \in U_i$ and $g_i(x_\lambda) - f_i(x_\lambda) \in U_i$. Further $x, x_\lambda \in X_\lambda$ implies $g_i(x_\lambda) = g_i(x)$. Hence we have $f_i(x_\lambda) - f_i(x) = \{g_i(x) - f_i(x)\} - \{g_i(x_\lambda) - f_i(x_\lambda)\} + \{g_i(x_\lambda) - g_i(x)\} \in U_i - U_i + 0 \subset V_i$. Thus the definition of V_i implies that $f(x_\lambda) - f(x) = \varphi(f_1(x_\lambda), \cdots, f_n(x_\lambda)) - \varphi(f_1(x), \cdots, f_n(x)) \in V$. Since $g(x) = f(x_\lambda)$ and since $x \in X$, we have $g(x) - (Xf)(x) = f(x_\lambda) - f(x) \in V$ and this proves the theorem.

Let us consider a fixed topological additive group K and discuss the relation between the group $\mathcal{G}(K)$ and the set $\mathcal{H}$ of all measurable maps of M into K.

Assumption 3.2. *K is a topological additive group.*

Let us denote $\Lambda(K)$, $\mathcal{F}(K)$, $\mathcal{G}(K)$, $\mathcal{G}_0(K)$, and $\mathcal{V}(K)$ by Λ, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{G}_0$, and $\mathcal{V}$, respectively. Further let us denote by $\mathcal{H}$ the set of all measurable maps of M into K.

The following lemma is easily verified:

Lemma 3.1. *For any $X \in \mathcal{S}$, $f \in \mathcal{F}$, and for any subset E of K, it holds that*

$$(Xf)^{-1}(E) = \begin{cases} f^{-1}(E) \cap X & (0 \notin E) \\ f^{-1}(E) \cup X^c & (0 \in E). \end{cases}$$

Then we can state an elementary property of $\mathcal{H}$:

Proposition 3.1. *It holds that*

1) *If $X \in \mathcal{S}$ and $f \in \mathcal{H}$, then $Xf \in \mathcal{H}$.*

2) *If $f \in \mathcal{F}$ and if $Xf \in \mathcal{H}$ for any $X \in \mathcal{S}$, then $f \in \mathcal{H}$.*

Proof. To prove 1), let O be an open set in K and Y an element of $\mathcal{S}$. For the case $0 \notin O$ we have $(Xf)^{-1}(O) \cap Y = (f^{-1}(O) \cap X) \cap Y = f^{-1}(O) \cap (XY) \in \mathcal{S}$, and for the case $0 \in O$ we have $(Xf)^{-1}(O) \cap Y$

$=(f^{-1}(O)\cup X^c)\cap Y=(f^{-1}(O)\cap Y)\cup(Y-X)\in\mathcal{S}$. Thus 1) is proved. For an open set O in K and for an element X in $\mathcal{S}$ we have $f^{-1}(O)\cap X=(Xf)^{-1}(O)\cap X\in\mathcal{S}$ and this implies 2).

Corollary. *If $\mathcal{H}$ is a subgroup of $\mathcal{F}$, then $\mathcal{H}$ is a $\mathcal{F}$-complete subgroup of $\mathcal{F}$.*

Put $\mathcal{B}_0=\{f\mid f\in\mathcal{F}$ and $f(M)$ is totally bounded$\}$ and $\mathcal{B}=\{f\mid f\in\mathcal{F}$ and $f(X)$ is totally bounded for any $X\in\mathcal{S}\}$. Then we have

Lemma 3.2. *$\mathcal{B}_0$ is an $\mathcal{S}$-invariant closed subgroup of $\mathcal{F}$ and $\mathcal{B}$ is the $\mathcal{F}$-completion of $\mathcal{B}_0$.*

Proof. It is easily seen that $\mathcal{B}_0$ is an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$. To prove that $\mathcal{B}_0$ is closed, let f be an element of $\overline{\mathcal{B}}_0$ and V an element of $\mathcal{V}$. Then we have $U\in\mathcal{V}$ such that $2U\subset V$. Since $f\in\overline{\mathcal{B}}_0$ we have $g\in\mathcal{B}_0$ such that $(f-g)(M)\subset U$ and $g\in\mathcal{B}_0$ implies the existence of $a_1,\cdots,a_n\in K$ such that $g(M)\subset\bigcup_{i=1}^n(a_i+U)$. Then for any $x\in M$ we have $f(x)=\{f(x)-g(x)\}+g(x)\in U+\bigcup_{i=1}^n(a_i+U)\subset\bigcup_{i=1}^n(a_i+2U)\subset\bigcup_{i=1}^n(a_i+V)$. Thus we have $f\in\mathcal{B}_0$ and this implies that $\mathcal{B}_0$ is closed. That $\mathcal{B}$ is the $\mathcal{F}$-completion of $\mathcal{B}_0$ is easily verified.

Corollary. *$\mathcal{B}$ is an i-closed subgroup of $\mathcal{F}$.*

Proof. This follows from Proposition 3.17 in [2].

Now we have

Theorem 3.2. *It holds that $K\cup\mathcal{G}_0\subset\mathcal{H}\cap\mathcal{B}\subset\mathcal{G}\subset\mathcal{B}$.*

Proof. The first inclusion is easily verified and the last one follows from Corollary to Lemma 3.2 and the fact that $\mathcal{G}_0\subset\mathcal{B}$. The second inclusion follows immediately from the following lemma.

Lemma 3.3. *Let $\mathcal{J}$ be the set of all $f\in\mathcal{F}$ satisfying the condition: for any $V\in\mathcal{V}$ and $X\in\mathcal{S}$, there exists $U_X\in\mathcal{V}$ such that for any $x\in X$ it holds that $f^{-1}(f(x)+E_x)\cap X\in\mathcal{S}$ for some subset E_x of K satisfying $U_X\subset E_x\subset V$. Then we have*

1) $\mathcal{H}\subset\mathcal{J}$.

2) $\mathcal{J}\cap\mathcal{B}\subset\mathcal{G}$.

Proof. 1) Let f be an element of $\mathcal{H}$. For any $V\in\mathcal{V}$ and $X\in\mathcal{S}$ we have an open set O in K such that $0\in O\subset V$. Further we have $U\in\mathcal{V}$ such that $U\subset O$. Put $U_X=U$ and for each $x\in X$ put $E_x=O$. Then we have $U_X\subset E_x\subset V$ and $f^{-1}(f(x)+E_x)\cap X\in\mathcal{S}$. This implies $f\in\mathcal{J}$. 2) Let f be an element of $\mathcal{J}\cap\mathcal{B}$. Then it suffices to show that $Xf\in\overline{\mathcal{G}}_0$ for any $X\in\mathcal{S}$. For given $V\in\mathcal{V}$ we are to show the existence of $g\in\mathcal{G}_0$ such that $(Xf)(x)-g(x)\in V$ for any $x\in M$. Since $f\in\mathcal{J}$ we have $U_X\in\mathcal{V}$ satisfying the condition: for any $x\in X$ there exists a subset E_x of K such that $U_X\subset E_x\subset V$ and $f^{-1}(f(x)+E_x)\cap X\in\mathcal{S}$. Since $f\in\mathcal{B}$ we have $x_1,\cdots,x_n\in X$ such that $f(X)\subset\bigcup_{i=1}^n(f(x_i)+U_X)$. Putting $Y_i=f^{-1}(f(x_i)+E_{x_i})\cap X$ we have $Y_i\in\mathcal{S}$, $i=1,2,\cdots,n$. For any $x\in X$ there exists $k\leqq n$ such that $f(x)\in f(x_k)+U_X\subset f(x_k)+E_{x_k}$ and this implies that $x \in Y_k$. Thus we have $\bigcup_{i=1}^n Y_i = X$. Let us define X_i inductively by $X_i = Y_i - \bigcup_{j=1}^{i-1} Y_j$, $i=1,2,\cdots,n$. Then we have $X_i \in \mathcal{S}$ such that $X_i \subset Y_i$ for $i=1,2,\cdots,n$, and it follows that $X_j X_k = 0$ $(j \neq k)$ and $X = \sum_{i=1}^n X_i$. Thus putting $g = \sum_{i=1}^n X_i f(x_i)$ we have $g \in \mathcal{G}_0$. Let us prove that $(Xf)(x) - g(x) \in V$ for each $x \in M$. Since $(Xf)(x) = 0 = g(x)$ for $x \notin X$, we may assume that $x \in X$. Then $x \in X_l$ for some $l \leqq n$, and it follows from $X_l \subset Y_l$ that $x \in Y_l$. Hence we have $f(x) \in f(x_l) + E_{x_l} \subset f(x_l) + V$ and this implies that $(Xf)(x) - g(x) = f(x) - f(x_l) \in V$. Thus the lemma is proved.

Assumption 3.3. *$\mathcal{S}$ is a pseudo-σ-ring and K satisfies the first condition of countability.*

Under the above assumption, Theorem 2.2 in [1] implies

Theorem 3.3. *Let f_i, $i=1,2,\cdots$, be elements of $\mathcal{H}$ and suppose that f_i converges pointwise to an element f of $\mathcal{F}$. Then f is an element of $\mathcal{H}$.*

Corollary 1. *$\mathcal{H}$ is closed in $\mathcal{F}$.*

Proof. Let f be an element of $\overline{\mathcal{H}}$. Assumption 3.3 implies that $\mathcal{F}$ satisfies the first condition of countability. Thus we have $f_i \in \mathcal{H}$, $i=1,2,\cdots$, such that $f_i \to f$ $(i \to \infty)$ in $\mathcal{F}$. Since f_i converges pointwise to f, the theorem implies $f \in \mathcal{H}$.

Corollary 2. *If K is completely separable, then $\mathcal{H}$ is an i-closed subgroup of $\mathcal{F}$.*

Proof. This follows from the above corollary, Corollary 2 to Theorem 2.1 in [1], and Corollary to Proposition 3.1.

Corollary 3. *It holds that $\mathcal{H} \cap \mathcal{B} = \mathcal{G}$.*

Proof. Let f be an element of $\mathcal{G}$. Then, for any $X \in \mathcal{S}$, we have $Xf \in \overline{\mathcal{G}}_0 \subset \overline{\mathcal{H}} = \mathcal{H}$. Hence Proposition 3.1 implies $f \in \mathcal{H}$. Thus we have $\mathcal{G} \subset \mathcal{H}$. Hence our corollary follows from Theorem 3.2.

References

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