7. On Measurable Functions. I

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1. Introduction. An integral structure Γ was defined and an integral σ with respect to Γ was discussed in the author [3]. Let $\Lambda=(M, G, K, J)$ be an integral system and $\mathcal{S}$ a measurable ring of Λ. Then the fundamental integral structure $\Gamma=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ is determined by Λ and $\mathcal{S}$. Theorem 1 in [3] states that there exists a unique integral with respect to Γ provided that J is Hausdorff and complete. The set $\mathcal{G}$ of all integrands is the integral closure of K in the total functional group $\mathcal{F}$ of Λ with respect to the abstract integral structure $(\mathcal{S}, \mathcal{F}, J)$.

In this part of the paper, we shall define the measurability of a function $f \in \mathcal{F}$ and state some properties of measurable functions. Some relations between the set $\mathcal{M}$ of all measurable functions and the set $\mathcal{G}$ of all integrands will be discussed in Part II.

2. Measurable functions. Assumption 2.1. *M is a set and $\mathcal{S}$ is a ring of subsets of M.*

A map f of M into a topological space K is *measurable* if $f^{-1}(O) \cap X \in \mathcal{S}$ for any open set O in K and for any $X \in \mathcal{S}$.

Proposition 2.1. *Let N be a set and $\mathcal{A}$ a set of subsets of N. Let f be a map of M into N such that $f^{-1}(Y) \cap X \in \mathcal{S}$ for any $Y \in \mathcal{A}$ and $X \in \mathcal{S}$. Then we have*

1) *$f^{-1}(Y) \cap X \in \mathcal{S}$ for any element Y of the ring generated by $\mathcal{A}$ and for any $X \in \mathcal{S}$.*

2) *Assume that $\mathcal{S}$ is a pseudo-σ-ring. Then $f^{-1}(Y) \cap X \in \mathcal{S}$ for any element Y of the σ-ring generated by $\mathcal{A}$ and for any $X \in \mathcal{S}$.*

Proof. Putting $\mathcal{T}=\{Y \mid Y \subset N, f^{-1}(Y) \cap X \in \mathcal{S}$ for any $X \in \mathcal{S}\}$, we have $\mathcal{A} \subset \mathcal{T}$. For $Y, Z \in \mathcal{T}$ and for any $X \in \mathcal{S}$, it holds that $f^{-1}(Y-Z) \cap X=(f^{-1}(Y)-f^{-1}(Z)) \cap X=(f^{-1}(Y) \cap X)-(f^{-1}(Z) \cap X) \in \mathcal{S}$ and hence $Y-Z \in \mathcal{T}$. Analogously, $Y \cup Z \in \mathcal{T}$ for any $Y, Z \in \mathcal{T}$. Since $\phi \in \mathcal{T}$, it follows that $\mathcal{T}$ is a ring containing $\mathcal{A}$. Hence $\mathcal{T}$ contains the ring generated by $\mathcal{A}$ and thus 1) is proved. If $\mathcal{S}$ is a pseudo-σ-ring, we have $\bigcup_{i=1}^{\infty} Y_i \in \mathcal{T}$, for $Y_i \in \mathcal{T}$, $i=1, 2, \cdots$, and this implies that $\mathcal{T}$ is a σ-ring containing $\mathcal{A}$. Thus 2) is proved.

Corollary 1. *Let K be a topological space and suppose that a map f of M into K is measurable. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be the ring and the σ-ring, respectively, generated by the set of all open sets in K. Then we have*

1) *$f^{-1}(Y) \cap X \in \mathcal{S}$ for any $Y \in \mathcal{T}_0$ and $X \in \mathcal{S}$.*

2) *If $\mathcal{S}$ is a pseudo-σ-ring, $f^{-1}(Y)\cap X\in\mathcal{S}$ for any $Y\in\mathcal{T}_1$ and $X\in\mathcal{S}$.*

Corollary 2. *Let K be a topological space and f a measurable map of M into K. Let K' be a topological space and g a map of K into K'. Then a sufficient condition for the composite map $g\circ f$ to be measurable is that one of the following conditions is satisfied:*

1) *For any open set O in K', $g^{-1}(O)$ is an element of the ring generated by the set of all open sets in K.*

2) *$\mathcal{S}$ is a pseudo-σ-ring. For any open set O in K', $g^{-1}(O)$ is an element of the σ-ring generated by the set of all open sets in K.*

3) *g is continuous.*

Proof. Let us prove that 1) and 2) are sufficient. Let O be an open set in K' and X an element of $\mathcal{S}$. Then it suffices to show that $(g\circ f)^{-1}(O)\cap X\in\mathcal{S}$. This follows from Corollary 1 and the fact that $(g\circ f)^{-1}(O)=f^{-1}(g^{-1}(O))$. The sufficiency of 3) follows immediately from that of 1).

To prove Theorem 2.1 below, we shall use the following topological lemma.

Lemma 2.1. *Let D be a completely separable topological space. Then any open base $\mathcal{O}$ of D contains a countable open base of D.*

Proof. Let $\mathcal{O}'=\{O_i\mid i=1,2,\cdots\}$ be a countable open base of D. Put $L=\{(l,m)\mid O_l\subset O\subset O_m$ for some $O\in\mathcal{O}\}$ and let O_{lm} be a fixed element of $\mathcal{O}$ such that $O_l\subset O_{lm}\subset O_m$ for each $(l,m)\in L$. Then $\mathcal{O}''=\{O_{lm}\mid (l,m)\in L\}$ is a countable open base of D contained in $\mathcal{O}$. In fact, for any open set Q and for any $p\in Q$, we have an element O_m of $\mathcal{O}'$ such that $p\in O_m\subset Q$. Further we have an element O of $\mathcal{O}$ such that $p\in O\subset O_m$. For an element O_l of $\mathcal{O}'$ such that $p\in O_l\subset O$, we have $(l,m)\in L$ and $p\in O_{lm}\subset Q$.

Assumption 2.2. *$\mathcal{S}$ is a pseudo-σ-ring.*

Theorem 2.1. *Let K_i be a topological space and f_i a measurable map of M into K_i for each $i\in I$, where I is a finite set $\{1,2,\cdots,n\}$ or a countable set $\{1,2,\cdots,n,\cdots\}$. Let $K=\prod_{i\in I}K_i$ be the product space of $K_i, i\in I$, with the strong (or weak) topology. Let us define a map f of M into K by $f(x)=(f_1(x),f_2(x),\cdots)$ for each $x\in M$. Then a sufficient condition for f to be measurable is that for each $X\in\mathcal{S}$ the subspace $f(X)$ of K is completely separable.*

Proof. Let O be an open set in K and X an element of $\mathcal{S}$. Then it suffices to show that $f^{-1}(O)\cap X\in\mathcal{S}$. Put $D=f(X)$. Since $\{D\cap\prod_{i\in I}O^i\mid O^i$ is an open set in K_i (and, if K has the weak topology, $O^i=K_i$ except finite i's)$\}$ forms an open base of D and since D is completely separable, Lemma 2.1 implies that D has an open base of the form $\{D\cap O_j\mid j=1,2,\cdots\}$, where $O_j=\prod_{i\in I}O_j^i$ with O_j^i an open set in K_i for each i and j. The measurability of f_i implies $f_i^{-1}(O_j^i)\cap X\in\mathcal{S}$. Since

$\mathcal{S}$ is a pseudo-σ-ring, we have $Y_j=(\bigcap_{i\in I} f_i^{-1}(O_j^i))\cap X=\bigcap_{i\in I}(f_i^{-1}(O_j^i)\cap X)\in\mathcal{S}$ for each j. Put $L=\{l\mid D\cap O_l\subset O\}$ and $Y=\bigcup_{l\in L} Y_l$. Then $Y_l\subset X$ implies $Y\in\mathcal{S}$. Hence it suffices to verify that $f^{-1}(O)\cap X=Y$. Suppose that $y\in Y$. Then we have $y\in Y_l=(\bigcap_{i\in I} f_i^{-1}(O_l^i))\cap X$ for some $l\in L$. This implies that $y\in X$ and $f_i(y)\in O_l^i$ for each $i\in I$ and hence $f(y)=(f_1(y),f_2(y),\cdots)\in D\cap O_l\subset O$. Thus it follows that $y\in f^{-1}(O)\cap X$. Conversely suppose that $y\in f^{-1}(O)\cap X$. Since $f(y)\in D\cap O$ and O is open, we have j such that $f(y)\in D\cap O_j\subset O$. Then it holds that $j\in L$ and $(f_1(y), f_2(y),\cdots)=f(y)\in O_j=\prod_{i\in I} O_j^i$. This implies that $y\in\bigcap_{i\in I} f_i^{-1}(O_j^i)$ and hence $y\in X$ implies that $y\in Y_j$. Since $j\in L$ we have $y\in\bigcup_{l\in L} Y_l=Y$. This completes the proof.

Corollary 1. *Let K_i be a topological space and f_i a measurable map of M into K_i for each $i=1,2,\cdots,n$. Let D be a completely separable subspace of $\prod_{i=1}^n K_i$ and φ a continuous map of D into a topological space K. Suppose that $(f_1(x),\cdots,f_n(x))\in D$ for each $x\in M$ and define a map f of M into K by $f(x)=\varphi(f_1(x),\cdots,f_n(x))$ for each $x\in M$. Then the map f is measurable.*

Proof. Define a map g of M into D by $g(x)=(f_1(x),\cdots,f_n(x))$ for each $x\in M$. Then Theorem 2.1 implies that g is measurable. Since $f=\varphi\circ g$, this corollary follows from Corollary 2 to Proposition 2.1.

Corollary 2. *Let K be a topological additive group and assume that K is completely separable. Then the set of all measurable maps of M into K is a subgroup of the additive group of all maps of M into K.*

Proof. Define a map φ of $D=K\times K$ into K by $\varphi(u,v)=u-v$. Then φ is continuous. Since K is completely separable, so is $D=K\times K$. Now let f and g be measurable maps of M into K. Then the map $f-g$ is the map h defined by $h(x)=\varphi(f(x),g(x))$ for each $x\in M$. Thus the measurability of $h=f-g$ follows from Corollary 1. Hence our corollary is proved.

The following topological lemma will be used to prove Theorem 2.2 below.

Lemma 2.2. *Let f and $f_i, i=1,2,\cdots$, be maps of M into a topological space K and suppose that f_i converges pointwise to f. Then, for any subset E of K and for $S(E)=\bigcup_{n=1}^\infty\bigcap_{i=n}^\infty f_i^{-1}(E)$, it holds that $f^{-1}(\mathring{E})\subset S(E)\subset f^{-1}(\bar{E})$.*[1]

Proof. For $x\in f^{-1}(\mathring{E})$, we have $f(x)\in\mathring{E}$ and hence there exists an n such that $f_i(x)\in\mathring{E}\subset E$ for any $i\geqq n$. This implies $x\in S(E)$. Now suppose that $y\in S(E)$. Then it suffices to show that $V\cap E\neq\phi$ for any neighbourhood V of $f(y)$. Since $f_i(y)$ converges to $f(y)$, we have an n_1 such that $f_i(y)\in V$ for any $i\geqq n_1$. Further $y\in S(E)$ implies the existence of an n_2 such that $f_i(y)\in E$ for any $i\geqq n_2$. For $n=\max(n_1,n_2)$ it

1) $\mathring{E}$ and $\bar{E}$ mean the interior and the closure, respectively, of E in K.

follows that $f_n(y) \in V \cap E$ and thus the lemma is proved.

Theorem 2.2. *Let K be a uniform space and suppose that K has a countable base for uniformity.*[1] *Let $f_i, i=1,2,\cdots$, be measurable maps of M into K and suppose that f_i converges pointwise to a map f of M into K. Then f is measurable.*

Proof. Let O be an open set in K and X an element of $\mathcal{S}$. Then it suffices to prove that $f^{-1}(O) \cap X \in \mathcal{S}$. Let $\{U_k \mid k=1,2,\cdots\}$ be a countable base for uniformity of K. We may assume that $U_{k+1}^3 \subset U_k = U_k^{-1}$ for each k. Putting $O_k = (\overline{\bigcup_{t \in O^c} U_k(t)})^c$ we have open sets $O_k, k=1,2,\cdots$, in K. Let us prove 1) $\bar{O}_k \subset O_{k+1}$ for each k and 2) $O = \bigcup_{k=1}^{\infty} O_k$. 1) is proved as follows. For any $x \in \bar{O}_k$ there exists $y \in U_{k+1}(x) \cap O_k$. Suppose that $x \notin O_{k+1}$. Since $x \in O_{k+1}^c = \overline{\bigcup_{t \in O^c} U_{k+1}(t)}$ we have $t_0 \in O^c$ and $z \in U_{k+1}(x) \cap U_{k+1}(t_0)$. Hence $y \in U_{k+1}(x) \subset U_{k+1}^2(z) \subset U_{k+1}^3(t_0) \subset U_k(t_0) \subset \bigcup_{t \in O^c} U_k(t) \subset O_k^c$ and this is a contradiction. Thus we have $x \in O_{k+1}$ for any $x \in \bar{O}_k$. Let us prove 2). $O_k^c \supset \bigcup_{t \in O^c} U_k(t) \supset O^c$ implies that $O_k \subset O$ for each k. Let us show that $x \in \bigcup_{k=1}^{\infty} O_k$ for any $x \in O$. There exists k such that $U_k(x) \subset O$ and it suffices to show that $x \in O_{k+1}$. Otherwise we have $x \in \overline{\bigcup_{t \in O^c} U_{k+1}(t)}$ and hence there exist $t_1 \in O^c$ and $y \in U_{k+1}(x) \cap U_{k+1}(t_1)$. This implies $t_1 \in U_{k+1}(y) \subset U_{k+1}^2(x) \subset U_k(x) \subset O$, which is a contradiction. Thus 1) and 2) are proved. Now put $S_k = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} f_i^{-1}(O_k)$ for each k. Then Lemma 2.2 implies that $f^{-1}(O_k) \subset S_k \subset f^{-1}(\bar{O}_k)$ for any k. Thus it follows that $f^{-1}(O) = f^{-1}(\bigcup_{k=1}^{\infty} O_k) = \bigcup_{k=1}^{\infty} f^{-1}(O_k) \subset \bigcup_{k=1}^{\infty} S_k \subset \bigcup_{k=1}^{\infty} f^{-1}(\bar{O}_k) = f^{-1}(\bigcup_{k=1}^{\infty} \bar{O}_k) = f^{-1}(\bigcup_{k=1}^{\infty} O_k) = f^{-1}(O)$. Hence we have $f^{-1}(O) \cap X = (\bigcup_{k=1}^{\infty} S_k) \cap X = \bigcup_{k=1}^{\infty} (S_k \cap X)$. Since $S_k \cap X = (\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} f_i^{-1}(O_k)) \cap X = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (f_i^{-1}(O_k) \cap X)$, since the measurability of f_i implies $f_i^{-1}(O_k) \cap X \in \mathcal{S}$, and since $\mathcal{S}$ is a pseudo-σ-ring, we have $f^{-1}(O) \cap X = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (f_i^{-1}(O_k) \cap X) \in \mathcal{S}$. Thus the theorem is proved.

References

[1] M. Takahashi: Integration with respect to the generalized measure. I, II. Proc. Japan Acad., **43**, 178–185 (1967).

[2] ——: Integration with respect to the generalized measure. III. Proc. Japan Acad., **44**, 452–456 (1968).

[3] ——: Integration with respect to the generalized measure. IV. Proc. Japan Acad., **44**, 457–461 (1968).

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1) A *base for uniformity* is a non-empty set $\mathcal{U}$ of subsets of $K \times K$ subject to the condition: for each $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $\Delta \subset W^2 \subset U \cap V^{-1}$. Here $\Delta = \{(x,x) \mid x \in K\}$, $T^n = \{(x_0, x_n) \mid (x_{i-1}, x_i) \in T, i=1,2,\cdots,n\}$ for $T \subset K \times K$ and for a positive integer n, and $T^{-1} = \{(y,x) \mid (x,y) \in T\}$ for $T \subset K \times K$. For each $x \in K$, the set $\{t \mid (t,x) \in U\}$ is denoted by $U(x)$ for $U \subset K \times K$ and the set $\{U(x) \mid U \in \mathcal{U}\}$ forms a base of the system of neighbourhoods of x in the (underlying) topological space K.