25. *On Some Examples of Non-normal Operators.* II

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1. Introduction. Consider a (bounded linear) operator T acting on a Hilbert space $\mathfrak{H}$. As usual, cf. [3], we shall call

$$W(T)=\{(Tx\mid x)\,;\,\|x\|=1,x\in\mathfrak{H}\}$$

the *numerical range* of T. An operator T is called a *convexoid* if $\overline{W}(T)=\operatorname{co}\sigma(T)$, where $\overline{W}(T)$ is the closure of $W(T)$, $\sigma(T)$ is the spectrum of T and $\operatorname{co}M$ is the convex hull of a set M in the complex plane. We shall also say that T satisfies the *condition* (G_1) (in symbol, $T\in(G_1)$) if

$$\|(T-\lambda)^{-1}\|\leqq\frac{1}{\operatorname{dist}(\lambda,\sigma(T))} \tag{1}$$

for any $\lambda\notin\sigma(T)$. If $T\in(G_1)$, then T is a convexoid, cf. [1] and [7].

In a recnt paper [4], Luecke introduced a new class of operators: $T\in\mathfrak{R}$ if

$$\|(T-\lambda)^{-1}\|=\frac{1}{\operatorname{dist}(\lambda,W(T))} \tag{2}$$

for any $\lambda\notin\overline{W}(T)$. He proved the following theorem:

Theorem A (Luecke). *$T\in\mathfrak{R}$ if and only if $\partial W(T)\subset\sigma(T)$, where ∂M is the boundary of M.*

Luecke's definition and theorem are interested in their own right; they establish a closed connection between a growth condition of resolvents and a spectral property of operators. However, in the light of the theory of seminormal operators, Luecke's class $\mathfrak{R}$ is rather restrictive. Even in the case of finite dimensional spaces, $\mathfrak{R}$ consists of the multiples of the identity, so that general normal operators are excluded by $\mathfrak{R}$.

In the present note, we shall introduce a class of operators which is defined by a growth condition and includes both (G_1) and $\mathfrak{R}$. For this purpose, we need to define the *hen-spectrum* $\tilde{\sigma}(T)$ of an operator T by $\tilde{\sigma}(T)=([\sigma(T)^c]_\infty)^c$ where M^c is the complement of M and $[M]_\infty$ the component of the infinity (unbounded component) of M. Clearly, $[M]_\infty$ is unique if M is bounded. By the definition, it is clear that $\tilde{\sigma}(T)$ is a compact set in the plane and contains $\sigma(T)$. Furthermore, we need the following idea due to Saito [6]: T is called an operator satisfying the *condition* (G_1) *for* M if

$$\|(T-\lambda)^{-1}\|\leqq\frac{1}{\operatorname{dist}(\lambda,M)} \tag{3}$$

for any $\lambda \notin M$, where M is a closed set containing $\sigma(T)$. Particularly, we shall say that T satisfies the *condition* (H_1) (in symbol, $T \in (H_1)$) if T satisfies the condition (G_1) for $\tilde{\sigma}(T)$, that is,

$$(4) \qquad \|(T-\lambda)^{-1}\| \leqq \frac{1}{\operatorname{dist}(\lambda, \tilde{\sigma}(T))}$$

for any $\lambda \notin \tilde{\sigma}(T)$.

In this note, we shall construct operators satisfying the condition (H_1) in § 2 and apply them to study some relations between classes of non-normal operators in § 3. In § 4, we shall give two remarks on Luecke's principle of constructions of operators and his class $\mathfrak{R}$.

2. Construction. We shall use Luecke's principle to construct operators satisfying the condition (H_1), cf. also § 4.

Theorem 1. *If A is an operator and B is a normal operator with $\overline{W}(A) \subset \tilde{\sigma}(B)$, then $T = A \oplus B$ satisfies (H_1).*

Proof. By the hypothesis, we have

$$\tilde{\sigma}(T) \supset \tilde{\sigma}(A) \cup \tilde{\sigma}(B) = \tilde{\sigma}(B).$$

Consequently, for any $\lambda \notin \tilde{\sigma}(T)$, we have

$$\begin{aligned}\|(T-\lambda)^{-1}\| &= \max\left[\|(A-\lambda)^{-1}\|, \|(B-\lambda)^{-1}\|\right] \\ &\leqq \max\left[\frac{1}{\operatorname{dist}(\lambda, \overline{W}(A))}, \frac{1}{\operatorname{dist}(\lambda, \sigma(B))}\right] \\ &\leqq \max\left[\frac{1}{\operatorname{dist}(\lambda, \overline{W}(A))}, \frac{1}{\operatorname{dist}(\lambda, \tilde{\sigma}(B))}\right] \\ &= \frac{1}{\operatorname{dist}(\lambda, \tilde{\sigma}(B))} \\ &\leqq \frac{1}{\operatorname{dist}(\lambda, \tilde{\sigma}(T))},\end{aligned}$$

so that $T \in (H_1)$.

Before to proceed further, we shall look at an elementary property of the hen-spectra of operators:

Proposition 2. *$\tilde{\sigma}(T) \subset \operatorname{co} \sigma(T)$ for any T. Therefore, we have $\tilde{\sigma}(T) \subset \overline{W}(T)$.*

Proof. Since $\operatorname{co} \sigma(T)$ is connected, we have

$$(\operatorname{co} \sigma(T))^c = [(\operatorname{co} \sigma(T)^c]_\infty \subset [\sigma(T)^c]_\infty.$$

Hence we have

$$\operatorname{co} \sigma(T) \supset ([\sigma(T)^c]_\infty)^c = \tilde{\sigma}(T).$$

By Theorem A and Proposition 2, we have an another characterization of operators belonging to $\mathfrak{R}$:

Theorem 3. *$T \in \mathfrak{R}$ if and only if $\overline{W}(T) = \tilde{\sigma}(T)$.*

Proof. If $T \in \mathfrak{R}$, then $\partial W(T) \subset \sigma(T)$ by Theorem A, so that $\sigma(T)$ includes the convex curve $\partial W(T)$. Hence we have $\overline{W}(T) \subset \tilde{\sigma}(T) \subset \overline{W}(T)$, or $\overline{W}(T) = \tilde{\sigma}(T)$. Conversely, if $\overline{W}(T) = \tilde{\sigma}(T)$, then we have $\partial W(T) = \partial \tilde{\sigma}(T) \subset \sigma(T)$. Hence, by Luecke's theorem, we have $T \in \mathfrak{R}$.

Now, we shall give an operator which has a finer property than that of Theorem 1:

Theorem 4. *If A is an operator and B is a normal operator with $\overline{W}(A)\subset\sigma(B)$ and $\tilde{\sigma}(B)\neq\mathrm{co}\,\sigma(B)$, then $T=A\oplus B\notin\mathcal{R}$ but $T\in(H_1)$.*

Proof. From Theorem 1, we have $T\in(H_1)$. We wish to show that $\overline{W}(T)\neq\tilde{\sigma}(T)$. We have

$$\tilde{\sigma}(T)=\tilde{\sigma}(A)\cup\tilde{\sigma}(B)=\tilde{\sigma}(B)\neq\mathrm{co}\,\sigma(B)=\overline{W}(B)=\overline{W}(T),$$

so that we have $T\notin\mathcal{R}$.

Example. Let U be the bilateral shift of multiplicity 1 and B a normal operator with $\sigma(B)=\{\lambda\,;|\lambda|=4,|\lambda-3|=5\}$. Put $A=U-3$ and $T=A\oplus B$. Then we have by [3 ; Prob. 68]

$$\overline{W}(A)=\{\lambda\,;|\lambda+3|\leqq 1\}.$$

Hence $\overline{W}(A)\subset\sigma(B)$ and

$$\overline{W}(T)=\{\lambda\,;|\lambda|=4,\mathrm{Re}\,\lambda\leqq 0\}.$$

On the other hand, it is clear that $0\notin\tilde{\sigma}(T)$. From Theorem 4, we have $T\notin\mathcal{R}$ and $T\in(H_1)$.

3. Application. The following two theorems indicate the position of the class (H_1) among the classes of seminormal operators:

Theorem 5. *If an operator T satisfies (G_1), then T satisfies (H_1) too.*

Proof. Comparing (4) with (1), we hare that $T\in(G_1)$ implies $T\in(H_1)$.

Theorem 6. *If an operator T satisfies (H_1), then T is a convexoid.*

Proof. It is known in [5] that T is a convexoid if and only if T satisfies the condition (G_1) for $\mathrm{co}\,\sigma(T)$ in the sense of Saito. Hence Proposition 2 implies that T is a convexoid if $T\in(H_1)$.

Theorem 7. *The class (H_1) properly contains the class (G_1).*

Proof. We shall construct $T\in(H_1)$ using Theorem 1. Put

$$A=\begin{pmatrix}0&2\\0&0\end{pmatrix}. \tag{5}$$

Then we have $\sigma(A)=\{0\}$ and $W(A)=D$ where D is the unit disk. Moreover, let U be the simple bilateral shift. Then, by [3 ; Prob. 68], we have $\sigma(U)=C$ and $\tilde{\sigma}(U)=D$ where C is the unit circle. Put $T=A\oplus U$. Then T satisfies (H_1) by Theorem 1. Clearly, we have $\sigma(T)=\{0\}\cup C$. Furthermore, we have

$$\left(A+\frac{1}{2}\right)^{-1}=2\begin{pmatrix}1&-4\\0&1\end{pmatrix}.$$

If we put

$$x=\begin{pmatrix}0\\1\end{pmatrix},$$

then we have

$$\left\|\left(A+\frac{1}{2}\right)^{-1}x\right\|=2\sqrt{16+1}>2.$$

If $T \in (G_1)$, then we have

$$2 < \left\| \left(A+\frac{1}{2}\right)^{-1} x \right\| \leqq \left\| \left(A+\frac{1}{2}\right)^{-1} \right\| \leqq \frac{1}{\operatorname{dist}(-1/2, \sigma(T))} = 2$$

and this contradiction proves the theorem.

Remark. If $T \in (H_1)$ and $\sigma(T)$ is a connected set or a finite set, then T satisfies the condition (G_1). Therefore, *in the case of finite dimensional spaces, the condition* (H_1) *coincides with the normality.*

The above remark gives us an example of convexoids which does not belong to (H_1), that is, a non-normal finite dimensional convexoid is the desired which is already known, cf. [2; Remark to Theorem 7]. Hence, *the class* (H_1) *is properly contained in the class of all convexoids.*

The operator T in the proof of Theorem 7 belongs to $\mathfrak{R}$ by Theorem 3. Hence we have

Theorem 8. *There is an operator in* $\mathfrak{R}$ *which does not satisfy the condition* (G_1).

On the other hand, we have

Theorem 9. *If* $T \in \mathfrak{R}$, *then* T *satisfies the condition* (H_1).

Proof. Suppose that $T \in \mathfrak{R}$. Then we have for any $\lambda \notin \overline{W}(T)$

$$\|(T-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \overline{W}(T))} = \frac{1}{\operatorname{dist}(\lambda, \hat{\sigma}(T))},$$

by Theorem 3. Hence, we have $T \in (H_1)$.

An operator T is called a *normaloid* if $\|T\| = r(T)$ where $r(T)$ is the spectral radius of T:

$$r(T) = \sup\{|\lambda|;\ \lambda \in \sigma(T)\}.$$

Also, T is called a *numeroid* if $W(T)$ is a spectral set for T in the sense of von Neumann, cf. [2]. In the remainder of this section, we shall discuss some relations between these classes and the class (H_1).

Proposition 10. *There are*

(i) *a normaloid which does not belong to* (H_1),

(ii) *a numeroid which does not belong to* (H_1),

(iii) *an operator in* (H_1) *which is not a normaloid, and*

(iv) *an operator in* (H_1) *which is not a numeroid.*

Proof. Since a normaloid T needs not a convexoid, cf. [3], Theorem 6 implies (i) at once. By [2; Remark to Theorem 7],

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

is a non-normal numeroid if the triangle with vertices $\lambda_1, \lambda_2, \lambda_3$ contains the unit disk D. Hence, by the above remark, (ii) is proved. Let A be as in (5) and $B = U$ where U is the simple unilateral shift. Then, by Theorem 1, $A \oplus B = T \in \mathfrak{R}$ so that $T \in (H_1)$ by Theorem 5. However, T is not a normaloid, since $\|T\| \geqq 2$ and $r(T) = 1$ by the fact that $\sigma(T)$

$=\sigma(A)\cup\sigma(B)=D$. Hence we have (iii). Finally, (iv) is false and $T\in(H_1)$ is automatically a numeroid, then T is a normaloid, which contradicts (iii).

4. Appendix. Here we shall give to remarks: The one concerns with an extension of Theorem 1 according to the line of Saito's generalized growth condition and the other with Luecke's class $\mathfrak{R}$.

The following theorem gives us a unified formulation of known results (compare Theorem 1 and [2; Theorem A]):

Luecke's principle. *If A is an operator, X a closed set in the plane with $\overline{W}(A)\subset X$ and B a normal operator with $\sigma(B)\subset X$, then $T=A\oplus B$ satisfies (G_1) for X in the sense of Saito.*

Since the proof is completely analogous to that of Theorem 1, we shall omit it.

In the above, $\overline{W}(A)\subset X$ is essential; we can not replace by $X\supset\sigma(A)$, as in the following

Proposition 11. *If A does not satisfy $(G_1)$for X which is a closed set with $\sigma(A)\subset X\subset\overline{W}(A)$ and $X\neq\overline{W}(A)$, then $T=A\oplus B$ does not satisfy (G_1) for X whenever B is a normal operator with $\sigma(B)\subset X$.*

Proof. By the hypothesis, we have a $\lambda\notin X$ such that

$$\|(A-\lambda)^{-1}\|>\frac{1}{\operatorname{dist}(\lambda, X)}.$$

Hence, we have

$$\begin{aligned}\|(T-\lambda)^{-1}\|&=\max[\|(A-\lambda)^{-1}\|, \|(B-\lambda)^{-1}\|]\\&=\max\left[\|(A-\lambda)^{-1}\|, \frac{1}{\operatorname{dist}(\lambda, \sigma(B))}\right]>\frac{1}{\operatorname{dist}(\lambda, X)}.\end{aligned}$$

Finally, we shall introduce a class of operators. Let Q be the set of all operators satisfying

$$\tilde{\sigma}(T)=\operatorname{co}\sigma(T). \tag{6}$$

This is equivalent to state that $T\in Q$ if and only if $\partial\tilde{\sigma}(T)$ is a convex curve. By Theorem 3, $T\in\mathfrak{R}$ implies $T\in Q$. In the converse direction, we shall show the following theorem which gives an another characterization of Luecke's class:

Theorem 12. *$\mathfrak{R}=C\cap Q$ where C is the set of all convexoids.*

Proof. $\mathfrak{R}\subset C$ is proved by Luecke [4] and $\mathfrak{R}\subset Q$ is clear by the above. Hence $\mathfrak{R}\subset C\subset Q$. Conversely, if $T\in C\cap Q$, then we have

$$\tilde{\sigma}(T)=\operatorname{co}\sigma(T)=\overline{W}(T),$$

so that we have $C\cap Q\subset\mathfrak{R}$ by Theorem 3.

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