24. On a Generalization of Adasch's Theorem

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N. Adasch [1] generalized Köthe's theorem [4] on the equicontinuous set of linear continuous mappings from (F)-space into (LF)-space. In this paper, we shall go a step further.

In the first part, we introduce the concept of the countably boundedness which generalizes the theorem ([1](10)b)⇒c)). Next, we generalize the theorem ([1](12)a)⇒b)) in the second part and the theorems ([1](10)c)⇒b)), ([1](12)b)⇒a)) in the third part.

Throughout this paper, terminology and notation are the same as in [3], if nothing otherwise is mentioned.

1. Definition 1. Let E be a locally convex separative topological linear space and A a subset of it. We say that A is countably bounded if, for any sequence $\{x_n\}$ of elements of A, there exists an absolutely convex bounded set B of E such that $\{x_n\}\subset E_B$. When E is countably bounded, E is said to be countably bounded space.

We have easily next seven propositions.

Proposition 1. *Any bounded subset of a locally convex separative topological linear space is countably bounded.*

Proposition 2. *Any finite union or sum of countably bounded subsets and any subset of a countably bounded set is countably bounded. Especially, any subspace of a countably bounded space is a countably bounded space for the induced topology.*

Proposition 3. *Any finite product of countably bounded spaces is countably bounded. A countably bounded space E is countably bounded for the topology such that E has the same dual E'.*

Proposition 4. *If a locally convex separative topological linear space E has the first countability property of Mackey* [5], [*Proposition* 12 *of this paper*], *then E is countably bounded. Especially, every metrizable locally convex topological linear space is countably bounded.*

Proposition 5. *Let E be a locally convex separative topological linear space and B an absolutely convex bounded set of E. Then E_B is a countably bounded space for the topology of E_B, and the induced topology.*

Proposition 6. *Let E be the locally convex separative topological linear space which is the union of a sequence of linear subspaces $\{E_n\}$. Then the following assertions are equivalent:*

(a) *For each countably bounded subset A of E, there exists the positive integer k such that $A \subset E_k$.*

(b) *For each bounded subset B of E, there exists the positive integer k such that $B \subset E_k$.*

Proposition 7. *Let E, F be locally convex separative topological linear spaces, and H a set of linear mappings from E into F such that, for each bounded set B of E, the set $H(B) = \{u(x) \mid u \in H, x \in B\}$ is bounded in F. If A is a countably bounded set in E, the set $H(A)$ is countably bounded in F.*

Let $L(E, F)$ be the linear space of linear continuous mappings from E into F. Each equicontinuous set of $L(E, F)$ satisfies the condition of H in Proposition 7, so we have the following corollaries.

Corollary 1. *Instead of the condition of H in Proposition 7, we suppose that H is an equicontinuous set of $L(E, F)$, then we have the same conclusion. Especially, the linear continuous image of a countably bounded set is countably bounded.*

Corollary 2. *The quotient space of a countably bounded space and the linear continuous image of a countably bounded space are countably bounded.*

Proposition 8. *Let E, F be locally convex separative topological linear spaces and F be the union of a sequence of linear subspaces $\{F_n\}$. Suppose that H is the set of linear mappings from E into F, defined in Proposition 7. If, for each bounded set B of F, there exists the positive integer k such that $B \subset F_k$, then, for each countably bounded set A of E, there exists the positive integer k such that $H(A) \subset F_k$.*

Proof. By Proposition 7, $H(A)$ is countably bounded in F, so by Proposition 6, we have the conclusion.

We have the following theorem as the corollary of Proposition 8 and this theorem is a generalization of Adasch's theorem ([1](10)b)$\Rightarrow$c)).

Theorem 1. *Let E be a countably bounded space, and F a locally convex separative topological linear space which is the union of a sequence of linear subspaces $\{F_n\}$. If, for each bounded set B of F, there exists the positive integer k such that $B \subset F_k$, then, for each equicontinuous set H of $L(E, F)$, there exists the positive integer k such that $H(E) \subset F_k$.*

Corollary 1. *We suppose that locally convex separative topological linear space E is the union of a sequence of linear proper subspaces $\{E_n\}$. If, for each bounded set B of E, there exists the positive integer k such that $B \subset E_k$, then E cannot be countably bounded.*

Proof. We suppose that E is countably bounded. Of course, the identity mapping i on E is a linear continuous mapping, so, by Theorem 1, there exists the positive integer k such that $i(E) = E \subset E_k$. This is the contradiction.

Corollary 2. $\mathscr{D}$ *(the space of infinitely differentiable functions with compact support) is not countably bounded.*

Proof. Because $\mathscr{D}$ is the union of the sequence of (F)-spaces $\{E_n\}$, each of which is a linear proper subspace of $\mathscr{D}$, and each bounded set is in some E_n.

Corollary 3. *If a locally convex separative topological linear space E is countably bounded and has a fundamental sequence of bounded sets (that is, E has the second countability property of Mackey* [5]), *then there exists a bounded barrel in E.*

Proof. We can consider that E has a fundamental sequence of bounded sets $\{B_n\}$ such that each B_n is an absolutely convex closed bounded set. Then E is the union of the sequence of normed spaces $\{E_{B_n}\}$ and for each bounded set B in E, there exists the positive integer n such that $B \subset E_{B_n}$. So, by Corollary 1, there exists the positive integer k such that $E = \bigcup_{n=1}^{\infty} nB_k$. Therefore B_k is a bounded barrel in E.

Corollary 4. *Let E be a locally convex separative topological linear space. Then the following assertions are equivalent:*

(a) *The topology of E is the infimum of a sequence of topologies of normed space.*

(b) *E is a countably bounded bornological space having a fundamental sequence of bounded sets.*

Proof. (a)$\Rightarrow$(b). As E is the linear continuous image of a countably bounded space, E is countably bounded. On the other hand, E is a bornological space having a fundamental sequence of bounded sets ([2] chap. III § 2 exercice 21) b)).

(b)$\Rightarrow$(a). From Corollary 3, there exists a bounded barrel in E. Hence we have our assertion ([2] chap. III § 2 exercice 21) b)).

2. Here we wish to generalize the theorem ([1](12) a)$\Rightarrow$b)).

Definition 2. Let E be a locally convex separative topological linear space and A a subset of it. We say that A is countably bounded in the sense of Mackey, if, for any sequence $\{B_n\}$ of bounded subsets contained in A, there exists an absolutely convex bounded set B such that $\{B_n\} \subset E_B$ and each B_n is bounded in E_B.

Cleary, we have next five propositions from Definition 2.

Proposition 9. *Any bounded set of a locally convex separative topological linear space is countably bounded in the sense of Mackey.*

Proposition 10. *Any finite union or sum of sets having countably boundedness in the sense of Mackey and any subset of a countably bounded set in the sense of Mackey is countably bounded in the sense of Mackey. Especially, any subspace of the space which is countably bounded in the sense of Mackey, is countably bounded in the sense of Mackey.*

Proposition 11. *Any finite product of the spaces, each of which is countably bounded in the sense of Mackey, is countably bounded in the sense of Mackey. A space which is countably bounded in the sense of Mackey, is countably bounded in the sense of Mackey for the topology such that E has the same dual E'.*

Proposition 12. *Let E be a locally convex separative topological linear space. The following statements are equivalent:*

(a) *E is countably bounded in the sense of Mackey.*

(b) *E has the first countability property of Mackey.*

Proposition 13. *If a set A is countably bounded in the sense of Mackey, A is countably bounded.*

Proposition 14. *Let E, F be locally convex separative topological linear spaces and F be the union of a sequence of locally convex separative topological linear spaces $\{F_n\}$. Suppose that H is a set of linear mappings from E into F such that, for each bounded set B of E, the set $H(B)$ is bounded in F. If, for each bounded set B of F, there exists the positive integer k such that $B \subset F_k$ and B is bounded in F_k, then, for each set A of E which is countably bounded in the sense of Mackey, there exists the positive integer k such that $H(A) \subset F_k$ and, for each bounded set B contained in A, $H(B)$ is bounded in F_k.*

Proof. We suppose that the statement were false. Let $\{n_i \mid i=1,2,\cdots\}$ be the set of positive integers such that $H(A) \cap F_{n_i} \neq \emptyset$, then there exists the sequence $\{B_{n_i} \mid i=1,2,\cdots\}$ of bounded sets contained in A such that $H(B_{n_i}) \not\subset F_{n_i}$ or $H(B_{n_i})$ is not bounded in F_{n_i}. As A is countably bounded in the sense of Mackey, there exists an absolutely convex bounded set B such that $\{B_{n_i}\} \subset E_B$ and each B_{n_i} is bounded in E_B. The property of H implies that $H(B)$ is bounded in F, $\{H(B_{n_i})\} \subset F_{\Gamma(H(B))}$ and each $H(B_{n_i})$ is bounded in $F_{\Gamma(H(B))}$. On the other hand, there is the positive integer n such that $\Gamma(H(B)) \subset F_n$ and $\Gamma(H(B))$ is bounded in F_n. It follows that $\{H(B_{n_i})\} \subset F_n$ and each $H(B_{n_i})$ is bounded in F_n. So there exists the positive integer k such that $n = n_k$, and we have that $H(B_{n_k}) \subset F_{n_k}$ and $H(B_{n_k})$ is bounded in F_{n_k}. This is a contradiction.

We have the following theorem as the corollary of Proposition 14 and this theorem is a generalization of Adasch's theorem ([1](12)a)$\Rightarrow$b)).

Theorem 2. *Let E be a bornological locally convex separative topological linear space having the first countability property of Mackey and let F be a locally convex separative topological linear space which is the union of a sequence of locally convex separative topological linear spaces $\{F_n\}$. If, for each bounded set B of F, there exists the positive integer k such that $B \subset F_k$ and B is bounded in F_k, then, for each equicontinuous set H of $L(E, F)$, there exists the positive integer k such* *that $H(E)\subset F_k$ and H is an equicontinuous set of $L(E,F_k)$.*

Proof. Proposition 14 implies that there exists the positive integer k such that $H(E)\subset F_k$ and, for each bounded set B of $E, H(B)$ is bounded in F_k. As E is bornological, H is an equicontinuous set of $L(E,F_k)$.

Corollary. *A locally convex separative topological linear space E is normable if and only if E is a bornological space having the first and the second countability property of Mackey.*

Proof. The necessity is trivial.

The sufficiency follows from Theorem 2 in the same way that Corollary 3 of Theorem 1 follows from Theorem 1.

3. Finally, we generalize the theorems ([1](10) c)$\Rightarrow$b)), ([1](12)b)$\Rightarrow$a)).

Propositon 15. *Let E,F be locally convex separative topological linear spaces and F be the union of a sequence of linear subspaces $\{F_n\}$. Suppose that H is the set of linear mappings from E into F such that, for each bounded set B in $E, H(B)$ is bounded in F. If, for each H above defined, there exists the positive integer k such that $H(E)\subset F_k$, then, for each bounded set B of F, there exists the positive integer k such that $B\subset F_k$.*

Proof. Let x' be a non-zero continuous linear functional on E,B be any bounded set of F. We define the linear continuous mapping u_b from E into F to make each element x of E correspond to the element $x'(x)b$ in F for the fixed element b in B. Let $H=\{u_b \mid b\in B\}$. As $\Gamma(B)$ is bounded in F, for each absolutely convex neighbourhood V of 0 in F, there exists the positive number λ such that $\lambda V\supset\Gamma(B)$. Let $U=\{x \mid |x'(x)|\leqq 1/\lambda\}$, then U is a neighbourhood of 0 in E, and, for each b of B and each x of $U, u_b(x)=x'(x)b\in(1/\lambda)\Gamma(B)\subset V$. So, H is an equicontinuous set of $L(E,F)$ and H satisfies the condition of this proposition. It follows that there exists the positive integer k such that $H(E)\subset F_k$. The other hand, $H(E)\supset B$, so $B\subset F_k$.

We have the following theorem as the corollary of this proposition and this theorem is a generalization of the theorem ([1](10)c)$\Rightarrow$b)).

Theorem 3. *Let E,F be locally convex separative topological linear spaces and F be the union of a sequence of linear subspaces $\{F_n\}$. If, for each equicontinuous set H of $L(E,F)$, there exists the positive integer k such that $H(E)\subset F_k$, then, for each bounded set B in F, there exists the positive integer k such that $B\subset F_k$.*

Proposition 16. *Let E,F be locally convex separative topological linear spaces and F be the union of a sequence of locally convex separative topological linear spaces $\{F_n\}$. Suppose that H is the set of linear mappings from E into F such that, for each bounded set B in*

$E, H(B)$ is bounded in F. If, for each H above defined, there exists the positive integer k such that $H(E) \subset F_k$ and, for each bounded set B in $E, H(B)$ is bounded in F_k, then, for each bounded set B in F, there exists the positive integer k such that $B \subset F_k$ and B is bounded in F_k.

Proof. By the proof of Proposition 15, for each bounded set B in F, an equicontinuous set H of $L(E, F)$ such that $B \subset H(E)$, is defined. By the assumption, there exists the positive integer k such that $H(E) \subset F_k$ and, for each bounded set B' in E, $H(B')$ is bounded in F_k. So $B \subset F_k$. Let a be an element of E such that $x'(a)=1$, then, for each neighbourhood V of 0 in F_k, there exists the positive number λ such that

$$\lambda H(a) = \lambda\{x'(a)b \mid b \in B\} \subset V, \qquad \text{i.e. } \lambda B \subset V.$$

It follows that B is bounded in F_k.

We have the following theorem as the corollary of this proposition and this theorem is a generalization of the theorem ([1](12)b)$\Rightarrow$a)).

Theorem 4. *Let E, F be locally convex separative topological linear spaces and F be the union of a sequence of locally convex separative topological linear spaces $\{F_n\}$. If, for each equicontinuous set H of $L(E, F)$, there exists the positive integer k such that $H(E) \subset F_k$ and H is an equicontinuous set of $L(E, F_k)$, then, for each bounded set B in F, there exists the positive integer k such that $B \subset F_k$ and B is bounded in F_k.*

References

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