23. *On Symmetric Spaces*

By Yoshio TANAKA

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1. Introduction. A. V. Arhangel'skii [1] has introduced the notion of symmetric spaces and given many interesting results in the theory for metrizability and so on. In this note, we shall discuss some properties concerning symmetric spaces: symmetrizability of subspaces, local properties, products of symmetric spaces and images of symmetric spaces under suitable maps.

We assume all spaces are Hausdorff and all maps are continuous and onto. We denote by 2^X the collection of all subsets of X, and abbreviate by $\{x_i\}$ a sequence $\{x_i\,; i=1,2,\cdots\}$.

2. Preliminaries. We begin by recording definitions of symmetric spaces and related spaces.

Definition 2.1. A space X is *symmetric* if there is a real valued, non-negative function d defined on $X\times X$ satisfying the conditions: (1) $d(x,y)=0$ whenever $x=y$, (2) $d(x,y)=d(y,x)$, (3) $A\subset X$ is closed in X whenever $d(x,A)>0$ for any $x\in X-A$.

If we replace the condition (3) by the following: For $A\subset X$, $x\in\overline{A}$ whenever $d(x,A)=0$, then such a space is called *semi-metric* [9].

A space X is *sequential* if $A\subset X$ is closed whenever $A\cap C$ is closed for every compact metric subset C of X [6].

A space X is *θ-refinable* if for each open covering $\mathfrak{U}$ of X, there is a sequence $\{\mathfrak{U}_i\}$ of open refinements of $\mathfrak{U}$ such that, for $x\in X$, there is an open covering $\mathfrak{U}_i$ which is finite at x [15].

As is well known, we have the relations between such spaces and other spaces as follows:

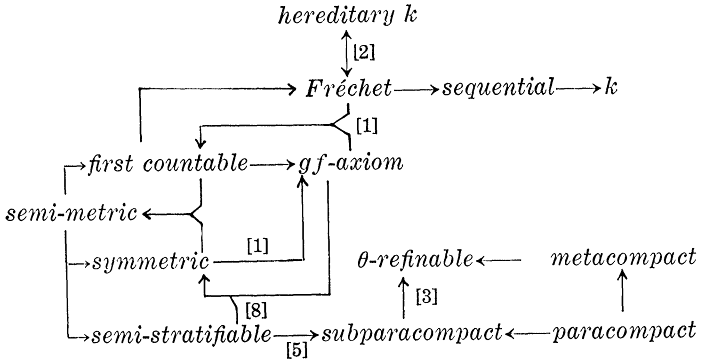

Fig. 2.2

Remark 2.3. By virtue of the above diagram we have the following theorem: *A space is semi-metric if and only if it is a hereditary symmetric space.* (We note that the proof of this theorem is directly given using the following Lemma 2.4.)

For later use, we state the following Lemma whose proof is straightforward and so is omitted (cf. the proof of [7; Theorem 3.2]).

Lemma 2.4.[*] *Let X be a space. Then the following are equivalent.*

(A): *X is a symmetric space,*

(B): *There is a sequence $\{g_i\}$ of functions from X into 2^X such that* (1) *$g_i(x) \ni x$,* (2) *$g_i(x) \ni y$ whenever $g_i(y) \ni x$, and* (3) *$O \subset X$ is open whenever, for each $x \in O$, there is i such that $g_i(x) \subset O$.*

(C): *X is a sequential space, and there is a sequence $\{g_i\}$ of functions from X into 2^X such that,* (1) *$g_i(x) \ni x$,* (2) *If $g_i(x) \ni x_i$, then $\{x_i\}$ converges to x,* (3) *If $g_i(x_i) \ni x$, then $\{x_i\}$ converges to x, and* (4) *For any first countable subset C containing x, $\mathrm{int}_C(g_i(x) \cap C) \ni x$ for each i.*

3. Subspaces and local properties. Theorem 3.1. *Let X be a symmetric space. Then a subspace A of X is symmetric if and only if it is a k-space.*

Proof. The "only if" part is obvious.

"if": Let $\{g_i\}$ be a sequence satisfying the conditions of (B) (of Lemma 2.4.). Define $(g_i|A)(x) = g_i(X) \cap A$ for $x \in A$. Then the sequence $\{g_i|A\}$ of functions from A into 2^A satisfies the conditions (1) and (2) of (B). Let $O \subset A$ and suppose that for $x \in O$, there is i such that $(g_i|A)(x) \subset O$. Assume O is not open in A. Since A is a k-space, there is a compact subset C of A such that $O \cap C$ is not open in C. For $x \in C$, let $(g_i|C)(x) = g_i(x) \cap C$. Then the sequence $\{g_i|C\}$ of functions from C into 2^C satisfies conditions of (B), because C is a closed subset of X. For $x \in O \cap C$, there is i such that $(g_i|A)(x) \subset O$. This implies $(g_i|C)(x) \subset O \cap C$. Thus $O \cap C$ is open in C, which is a contradiction. Hence O is open in A.

Conversely let O be an open subset of A. Then, for $x \in O$, there is i such that $(g_i|A)(x) \subset O$. Hence A is symmetric by Lemma 2.4.

The following example shows that a G_δ-set of a countable, symmetric space need not be symmetric.

Example 3.2 (cf. [6; Example 1.8]). Let $X = \{0\} \cup \bigcup_{i=1}^{\infty} X_i$, where $X_i = \{1/i\} \cup \{(1/i) + (1/j);\ j = 1, 2, \cdots\}$, be a space induced by the following function d on $X \times X$: For $x, x' \in X$, $d(x, x') = d(x', x)$, $d(0, 0) = 0$, and

$$d(x, x') = \begin{cases} |x - x'| & \text{if } x,\ x' \neq 0;\ \text{or } x = 0,\ x' = \dfrac{1}{i}. \\ 1 & \text{if } x = 0,\ 0 < x' \neq \dfrac{1}{i}. \end{cases}$$

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[*] (A)$\Longleftrightarrow$(B) has been pointed out by Takao Hoshina.

Since the function d satisfies the conditions of Lemma 2.1, X is a symmetric space. Let $A=X-\{1, 1/2, \cdots\}$. Then it is easy to see that a G_δ-set A of X is not a k-space. Then A is not symmetric.

In general, a countable CW-complex in the sense of J. H. Whitehead need not be a symmetric space. For example, let each I_i be a copy of the usual closed interval $[0,1]$, E be the topological sum of $\{I_i\}$, and $A=\{(0,i); i=1,2,\cdots\}$. Then $X=E/A$ is a countable CW-complex. Since X is Fréchet space, but is not first countable, it is not symmetric by Fig. 2.2.

But we have

Theorem 3.3. *Let a space X have the weak topology with respect to a point-finite covering $\{C_\alpha; \alpha \in A\}$, that is, a subset O of X is open whenever $O \cap C_\alpha$ is open in C_α for each $\alpha \in A$. If each C_α is symmetric, then X is symmetric.*

Proof. Since C_α is symmetric, there is a sequence $\{g_i^\alpha\}$ of functions from C_α into 2^{C_α} satisfying the conditions of (B). For $x \in C_\alpha$, we can assume $g_{i+1}^\alpha(x) \subset g_i^\alpha(x)$. Let $g_i(x)=\bigcup_{\alpha \in A(x)} g_i^\alpha(x)$, where $A(x) =\{\alpha; x \in C_\alpha\}$. Then it is easy to see that the sequence $\{g_i\}$ of functions from X into 2^X satisfies the conditions of (B). Hence X is a symmetric space by Lemma 2.4.

Corollary 3.4. *Let $\{C_\alpha; \alpha \in A\}$ be a point-finite open covering (or, a locally-finite closed covering) of X. If each C_α is symmetric, then X is a symmetric space.*

In general, a locally symmetric space need not be symmetric. For example, the space $X=[0, \Omega)$, where Ω is the first uncountable ordinal, with the order topology is countably compact and locally metric. Since a countably compact, symmetric space is compact [10], X is not symmetric. (We note that there is a Lindelöf, *Hausdorff*, locally metric space which is not symmetric.)

But from the following Lemma 3.5 and Lemma 2.4 (C), using the same method as in [13; Lemma 1.3] we have Theorem 3.6.

Lemma 3.5. *A locally sequential space is sequential.*

Theorem 3.6. *Let X be a θ-refinable space. If X is a locally symmetric space, then it is a symmetric space.*

4. Products of symmetric spaces. **Lemma 4.1** [11]. *Let X, Y be sequential spaces. Then $X \times Y$ is a sequential space if and only if it is a k-space.*

Theorem 4.2. *Let X, Y be symmetric spaces. Then $X \times Y$ is a symmetric space if and only if it is a k-space.*

Proof. The "only if" part is obvious.

"if": Since a symmetric space is sequential, by Lemma 4.1 $X \times Y$ is a sequential space. Then it is easy to see that $X \times Y$ has the weak topology with respect to $\{C\times K; C, K$ are compact metric subsets of X, Y respectively$\}$. Since X, Y are symmetric spaces, there are sequences $\{g_i^X\}$ and $\{g_i^Y\}$ satisfying the conditions of (C) with respect to X and Y respectively. We can assume $g_{i+1}^X(x)\subset g_i^X(x)$ and $g_{i+1}^Y(y)\subset g_i^Y(y)$. For $(x, y)\in X\times Y$, let $g_i(x, y)=g_i^X(x)\times g_i^Y(y)$. Then the sequence $\{g_i\}$ of functions from $X\times Y$ into $2^{X\times Y}$ satisfies the conditions of (B). Therefore $X\times Y$ is a symmetric space by Lemma 2.4.

Lemma 4.3 [4]. *Let X be a k-space and Y be a locally compact space. Then $X\times Y$ is a k-space.*

By Theorem 4.2 and Lemma 4.3, we have

Corollary 4.4. *Let X be a symmetric space and Y a locally compact, symmetric space. Then $X\times Y$ is a symmetric space.*

The following example shows that the product of a countable, symmetric space with a countable, metric space need not be symmetric.

Example 4.5. Let X be the space given in Example 3.2, and Y be X as a set. Define a function d on $Y\times Y$ as follows: $d(y, y')=0$ whenever $y=y'$ and $d(y, y')=M\text{ax}\{y, y'\}$ otherwise. Then a space Y induced by the function d is metric.

We shall prove the product $X\times Y$ is not a sequential space. (We note that $X\times X$ is a symmetric space.)
Choose pairwise disjoint, usual open intervals I_i containing $1/i$. Let $A=\{0\}\times Y\cup\{\bigcup_{i=1}^{\infty} V_i\cap(X\times Y)\}$, where

$$V_i=I_i\times\left\{y; y\leqq\frac{1}{i}\right\}\cup\left\{(x, y); y>2\left|x-\frac{1}{i}\right|+\frac{1}{i}\right\}.$$

Then $(0, 0)\in A-\text{int}\,A$. Hence A is not open in $X\times Y$. Assume $X\times Y$ is a sequential space. Then it is easy to see that $X\times Y$ has the weak topology with respect to $\{C\times K; C, K$ are convergent sequences on X, Y respectively$\}$. We remark that, if a sequence $\{x_i\}$ on X converges to the point zero, then we can assume the set $\{x_i: i=1, 2, \cdots\}$ is contained in $\{1, 1/2, \cdots\}$. Then we can prove that, for any convergent sequences C, K on X, Y respectively, $A\cap(C\times K)$ is always open in $C\times K$. Hence A is open in $X\times Y$, which is a contradiction. Thus $X\times Y$ is not sequential. Hence $X\times Y$ is not symmetric.

5. Images of symmetric spaces. Lemma 5.1 [12]. *Let $f: X\to Y$ be a closed map and each $f^{-1}(y)$ be first countable. If Y is first countable, then so is X.*

Theorem 5.2. *Let $f: X\to Y$ be an open-closed map, and each $f^{-1}(y)$ be first countable. If X is a symmetric space, then so is Y.*

Proof. Since X is a symmetric space, there is a sequence $\{h_i\}$ of functions from X into 2^X satisfying the conditions of (C). For $y\in Y$, choose a point x_y of $f^{-1}(y)$. Let $g_i'(y)=f(h_i(x_y))$. Then (1) $g_i'(y)\ni y$ and (2) If $g_i'(y)\ni y_i$, then $\{y_i\}$ clearly converges to y. Let $C\subset Y$ be a first countable subset containing y. Then $K=f^{-1}(C)$ is first countable by Lemma 5.1. Thus $x_y \in \operatorname{int}_K(h_i(x_y)\cap K)$ for each i. Since $f|K: K \to C$ is open, $y \in f(\operatorname{int}_K(h_i(x_y)\cap K)) \subset \operatorname{int}_C(g'_i(y)\cap C)$. Hence (3) $y \in \operatorname{int}_C(g'_i(y)\cap C)$ for any first countable subset C containing y.

Let $g''_i(y)=Y-f(X-\cup\{h_i(x); x \in f^{-1}(y)\})$. Then (4) $g''_i(y) \ni y$ and (5) If $g''_i(y_i) \ni y$, then it is easy to see that $\{y_i\}$ converges to y. Let $A=C-f(K-\cup \operatorname{int}_K(h_i(x)\cap K; x \in f^{-1}(y)))$. Then $y \in A \subset g''_i(y)\cap C$, and A is an open subset of C, because $f|K: K \to C$ is a closed map. Hence (6) $y \in \operatorname{int}_C(g''_i(y)\cap C)$ for any first countable subset C containing y.

Let $g_i(y)=g'_i(y)\cap g''_i(y)$. Then $g_i(y) \ni y$ by (1) and (4). If $g_i(y) \ni y_i$, then $\{y_i\}$ converges to y by (2). If $g_i(y_i) \ni y$, then $\{y_i\}$ converges to y by (5) For any first countable subset C of Y containing y, $y \in \operatorname{int}_C(g_i(y)\cap C)$ for each i by (3) and (6). Thus the sequence $\{g_i\}$ of functions from Y into 2^Y satisfies the conditions of (C).

On the other hand, Y is sequential by [6; Proposition 1.2]. Hence Y is a symmetric space by Lemma 2.4.

Since a compact, symmetric space is metric [1], we have

Corollary 5.3. *Let $f: X \to Y$ be an open, perfect map. If X is a symmetric space, then so is Y.*

Example 5.4. (1): The image of a countable, symmetric space under a perfect map need not be symmetric.

Let X be the space given in Example 3.2, $B=\{0\}\cup\{1, 1/2, \cdots\}$, and $Y=X/B$. Then the natural map $f: X \to Y$ is a perfect map. It is easy to see that Y is a Fréchet space, but is not first countable. Thus Y is not a symmetric space by Fig. 2.2.

(2): There is a *Hausdorff* space which is the image of a symmetric space under an open compact, countable-to-one map, but is not symmetric (cf. [14; Example 3.4.]).

We note that the image of a symmetric space under a closed (or, an open) finite-to-one map is always symmetric [14].

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