20. On the Existence of Hyperfunction Solutions of Linear Differential Equations of the First Order with Degenerate Real Principal Symbols

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0. Introduction. Let P be a first order linear partial differential operator of the form $P=\sum_{i,j=1}^{n} a_j^i x_j \partial/\partial x_i$, where $(a_j^i) \in GL(n, \boldsymbol{R})$. In this note we study the problem of existence of hyperfunction solutions of the equation $Pu=f$ mainly in the case where $n=2$, and show that the local solvability is valid in some cases. It is easy to see that P is locally solvable except in the neighborhood of the origin. On the other hand, the principal symbol of P vanishes at the origin. Hence our main interest is in the local solvability of the equation at the origin. By the flabbiness of the sheaf $\mathcal{B}$ of hyperfunctions we can deduce the local solvability from the global solvability, while we will show later that $P\mathcal{B}(\boldsymbol{R}^n)=\mathcal{B}(\boldsymbol{R}^n)$ in some cases.

H. Suzuki [4] has studied the local solvability of linear partial differential equations of the first order in two independent variables where the principal symbols do not vanish. He uses characteristic curves in the complex domain. Recently Suzuki [5] generalized Lemma 2 in Suzuki [4] as follows:

Theorem 0.1. *Let V be a domain of holomorphy and $P=\partial/\partial x_1$. Then the following are necessary and sufficient conditions for $P\mathcal{O}(V)=\mathcal{O}(V)$;*

(a) *For every $x \in V$, L_x is simply connected, where L_x is the connected component of the set $\{x' \in V; x'_i = x_i \text{ for } i=2, \cdots, n\}$ which contains the point x.*

(b) *The topology of V/P is Hausdorff, where V/P is the quotient space of V by the equivalence relation "$L_x = L_{x'}$" $(x, x' \in V)$.*

(c) *V/P is a domain of holomorphy over $\boldsymbol{C}^{n-1}$.*

We shall mainly use the same method as Suzuki [4] and rely on Theorem 0.1. But in one case we have to employ a different method (Case 1). The reason why we have to use two different methods may be explained by the theory of the sheaf $\mathcal{C}$. The author expresses his gratitude to Mr. Kashiwara for this suggestion. Lastly we remark that in the case of ordinary differential operators the problem has been completely solved by Sato [3] and Komatsu [1].

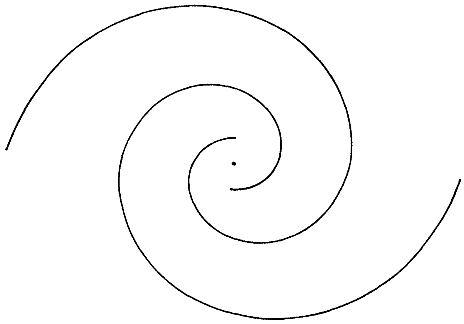
1. Canonical forms of operators under real coordinate transformations ($n=2, 3$). We have the following canonical forms of operators by real linear transformations. We use the notations x, y, z, in place of x_1, x_2, x_3 respectively, and use them to denote complex variables as well as real variables.

$n=2$	*Case* 1	$(\mu x-\nu y)\partial/\partial x+(\nu x+\mu y)\partial/\partial y$	$(\mu \neq 0, \nu \neq 0)$.
	Case 2	$y\partial/\partial x-x\partial/\partial y$.	
	Case 3	$x\partial/\partial x+\lambda y\partial/\partial y$	$(0<\lvert\lambda\rvert\leqq 1)$.
	Case 4	$(x+y)\partial/\partial x+y\partial/\partial y$.	
$n=3$	*Case* 1	$x\partial/\partial x+(\mu y-\nu z)\partial/\partial y+(\nu y+\mu z)\partial/\partial z$	$(\mu>0, \nu \neq 0)$.
	Case 2	$x\partial/\partial x+(\mu y-\nu z)\partial/\partial y+(\nu y+\mu z)\partial/\partial z$	$(\mu<0, \nu \neq 0)$.
	Case 3	$x\partial/\partial x+z\partial/\partial y-y\partial/\partial z$.	
	Case 4	$x\partial/\partial x+\lambda_1 y\partial/\partial y+\lambda_2 z\partial/\partial z$	$(0<\lvert\lambda_1\rvert, \lvert\lambda_2\rvert\leqq 1)$.
	Case 5	$(x+y)\partial/\partial x+y\partial/\partial y+\lambda z\partial/\partial z$	$(\lambda \neq 0)$.
	Case 6	$(x+y)\partial/\partial x+(y+z)\partial/\partial y+z\partial/\partial z$.	

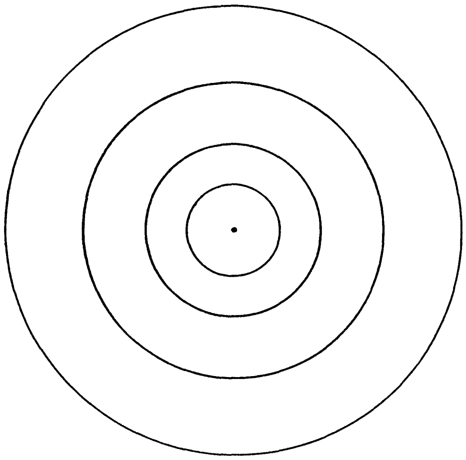
Where $n=2$ we can also write down the equations which define the bicharacteristic curves (Pontrjagin [2]).

Figures.

$n=2$ *Case* 1

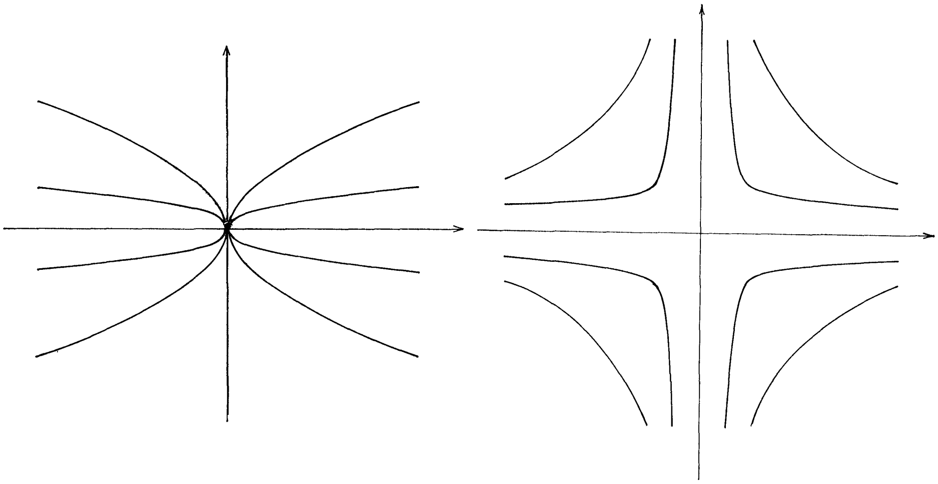


Case 2 $x^2+y^2=$constant.

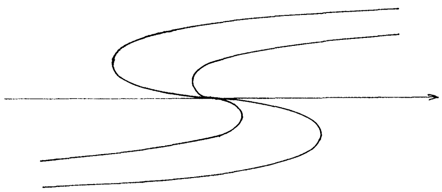


Case 3 $x^{-\lambda}y=\text{constant}$.

$0<\lambda\leqq 1$ $-1\leqq\lambda<0$



Case 4 $\log y-x/y=\text{constant}$.



2. Case 1 ($n=2, 3$). In this case we employ a different method from Suzuki's reasoning.

Lemma 2.1. *Let $\mathscr{B}[0]$ be the space of hyperfunctions with support at the origin. Then we have $P\mathscr{B}[0]=\mathscr{B}[0]$.*

Proof. We prove this lemma in the case where $n=3$. The case where $n=2$ is easier. Let $f\in\mathscr{B}[0]$. Then f has the standard defining function

$$F=\sum_{i,j,k=1}^{\infty} f_{ijk}\frac{1}{x^i y^j z^k} \quad \text{with} \quad \overline{\lim}^{\,i+j+k}\sqrt{|f_{ijk}|}=0.$$

We want to find $u\in\mathscr{B}[0]$ with the standard defining function

$$U=\sum_{i,j\,k=1}^{\infty} u_{ijk}\frac{1}{x^i y^j z^k}$$

such that $Pu=f$. Let f_{im} (respectively u_{im}) be the vector $(f_{i1m},\cdots,f_{im1})$ (respectively $(u_{i1m},\cdots,u_{im1})$). Then we have

$$(A_m-iI_m)u_{im}=f_{im}, \tag{2.1}$$

where

$$A_m=\begin{pmatrix} -(m+1)\mu & (m-1)\nu & & & & \\ -\nu & -(m+1)\mu & (m-2)\nu & & & \\ & -2\nu & -(m+1)\mu & (m-3)\nu & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \nu \\ & & & & -(m-1)\nu & -(m+1)\mu \end{pmatrix}.$$

Since $\mu>0$, the signatures of components of A_m-iI_m are

$$\begin{pmatrix} - & \pm & & & \\ \mp & - & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot\ - & \pm \\ & & & \mp & - \end{pmatrix}.$$

Hence it follows that $\det(A_m-iI_m)\neq 0$. Moreover we have

$$|\det(A_m-iI_m)|\geqq\{(m+1)\mu+i\}^m. \tag{2.2}$$

We shall prove $\overline{\lim}^{i+j+k}\sqrt{|u_{ijk}|}=0$ when we determine u_{ijk} by (2.1).

A little calculation shows that

$$|\det R^l_{im}|\leqq 3^{m-2}m\{(m+1)M+i\}^{m-1}|f_{im}|, \tag{2.3}$$

where R^l_{im} is the matrix obtained by replacing the l-th column of A_m-iI_m by f_{im}, $|f_{im}|=\max_p|f_{ipm-p}|$, $M=\max(\mu,|\nu|)$. From (2.2), (2.3) and Cramer's formula, we have

$$|u_{im}|\leqq\frac{3^{m-2}m\{(m+1)M+i\}^{m-1}}{\{(m+1)\mu+i\}^m}|f_{im}|\leqq K^m|f_{im}|,$$

for some $K>0$. It follows that $\overline{\lim}^{i+j+k}\sqrt{|u_{ijk}|}=0$ and this proves the existence of required u.

Proposition 2.2. *In Case* 1 ($n=2,3$) *we have* $P\mathscr{B}(\boldsymbol{R}^n)=\mathscr{B}(\boldsymbol{R}^n)$.

Proof. It is easy to see that $P\mathscr{B}(\boldsymbol{R}^n-\{0\})=\mathscr{B}(\boldsymbol{R}^n-\{0\})$. In fact for $f\in\mathscr{B}(\boldsymbol{R}^n-\{0\})$ such that $\operatorname{supp} f\cap\{x\in\boldsymbol{R}^n;x_1^2+\cdots+x_n^2=r^2\}=\phi$ we can uniquely solve the equation if we give any Cauchy data on this sphere, and any $f\in\mathscr{B}(\boldsymbol{R}^n-\{0\})$ can be represented as a sum of such functions. (See Case 1 of Fig. for $n=2$.) Because of the flabbiness of the sheaf $\mathscr{B}$, it is sufficient to show $P\mathscr{B}(\boldsymbol{R}^n)=\mathscr{B}[0]$. But this is the direct consequense of Lemma 2.1.

3. The case of canonical form $P=x\partial/\partial x+\sum_{i=1}^n\lambda_iy_i\partial/\partial y_i$ (λ_i real, $0<|\lambda_i|\leqq 1$).

Proposition 3.1. *Let* $P=x\partial/\partial x+\sum_{i=1}^n\lambda_iy_i\partial/\partial y_i$, *where* λ_i *is real and* $0<|\lambda_i|\leqq 1$. *Then we have* $P\mathscr{B}(\boldsymbol{R}^n)=\mathscr{B}(\boldsymbol{R}^n)$.

Proof. Let $U_{\tau\varepsilon_1\cdots\varepsilon_n}=\{(x,y_1,\cdots,y_n)\in\boldsymbol{C}^{n+1};\tau\operatorname{Im}x>0,\varepsilon_1\operatorname{Im}y_1>0,\cdots,\varepsilon_n\operatorname{Im}y_n>0\}$ where $\tau,\varepsilon_1,\cdots,\varepsilon_n=\pm1$. It is sufficient to show that $P\mathcal{O}(U_{\tau\varepsilon_1\cdots\varepsilon_n})=\mathcal{O}(U_{\tau\varepsilon_1\cdots\varepsilon_n})$, and we rely on Theorem 0.1. We prove this in the case where $\tau,\varepsilon_1,\cdots,\varepsilon_n=1$. Let U denote $U_{1\cdots1}$ below. Note that complex bicharacteristic curves of P in U is defined by $x^{-\lambda_1}y_1=a_1,\cdots,x^{-\lambda_n}y_n=a_n$, where $\log x=\log|x|+\sqrt{-1}\arg x$, $0<\arg x<\pi$. The change of variables $X=x$, $Y_1=x^{-\lambda_1}y_1, \cdots, Y_n=x^{-\lambda_n}y_n$ transforms U into V and P into $X\partial/\partial X$. Since $X\neq 0$ in V, we may apply Theorem 0.1 if we show the following two facts (a) and (b):

(a) $V(a_1, \cdots, a_n)=\{x\in \boldsymbol{C}; \operatorname{Im} x>0, \operatorname{Im} a_1x^{\lambda_1}>0, \cdots, \operatorname{Im} a_nx^{\lambda_n}>0\}$ is either void or connected and simply connected.

(b) Let π be the projection from $\boldsymbol{C}^{n+1}$ to $\boldsymbol{C}^n$ such that

$$\pi(x, y_1, \cdots, y_n)=(a_1, \cdots, a_n).$$

Then $\pi(U)$ is a domain of holomorphy.

Taking account of the fact that $0<|\lambda_i|\leqq 1$, we can easily verify (a). Now let us prove (b). $\log a_i=\log|a_i|+\sqrt{-1}(\arg y_i-\lambda_i \arg x)$ is well-defined and holomorphic in $\pi(U)$. Moreover the holomorphic mapping Log from $\pi(U)$ to $\boldsymbol{C}^n$ defined by

$$\operatorname{Log}(a_1, \cdots, a_n)=(\log a_1, \cdots, \log a_n)$$

is bijective. It is sufficient to show that $\operatorname{Log}(\pi(U))$ is a domain of holomorphy, but this is a convex domain. Thus (b) is proved, and at the same time the proof of Proposition 3.1 is completed.

4. Case 4 ($n=2$) and Case 5 ($n=3$).

Proposition 4.1. *In Cases* 4 ($n=2$) *and* 5 ($n=3$), *we have* $P\mathcal{B}(\boldsymbol{R}^n)=\mathcal{B}(\boldsymbol{R}^n)$.

Proof. We prove this proposition in the case where $n=3$. The case where $n=2$ is easier. The method of the proof is the same as Proposition 3.1. Note that in this case complex bicharacteristic curves in U are defined by

$$\log y-x/y=a, \qquad \log z-\lambda \log y=b. \tag{4.1}$$

The change of variables $X=\log y-x/y$, $Y=y$, $Z=\log z-\lambda\log y$ transforms U into V and P into $Y\partial/\partial Y$. Since $Y\neq 0$ in V, we may apply Theorem 0.1 if we show the following two facts (a) and (b):

(a) $V(a,b)=\{y\in\boldsymbol{C}; \operatorname{Im} y>0, \operatorname{Im}\{y(\log y-a)\}>0, 0<\arg(b+\lambda\log y)<\pi\}$ is either void or connected and simply connected.

(b) Let π be the projection from $\boldsymbol{C}^3$ to $\boldsymbol{C}^2$ such that $\pi(x, y, z)=(a, b)$. Then $\pi(U)$ is a domain of holomorphy.

When $0<\arg y<\pi$, $\operatorname{Im}\{y(\log y-a)\}>0$ is equivalent to saying

$$\log|y|>-\frac{\cos(\arg y)}{\sin(\arg y)}(\arg y-\operatorname{Im} a)+\operatorname{Re} a. \tag{4.2}$$

Hence we can easily verify (a). Now let us prove (b). Let $\pi(U)(a)=\{b\in\boldsymbol{C}; (a, b)\in\pi(U)\}$. If we show that $\pi(U)(a)$ is independent of a, the proof of (b) and that of Proposition 4.1 is completed. But this is a trivial consequence of (4.1) and (4.2).

5. Remarks.

Remark 5.1. It is not possible to solve the problem in Case 1 ($n=2, 3$) by the method in sections 3 and 4. If it were possible, for every hyperfunction f with the singular support in a quadrant there would exist a solution u of $Pu=f$ whose singular support is in the quadrant. Taking account of Figures Case 1, it is easy to see that this is impossible. In fact, there exists f the singularities of which must propagate in at least one direction on the bicharacteristic strip.

Remark 5.2. In Case 2 ($n=2$), local solvability is not valid. See Figures Case 2. In Cases 2, 3, 6 ($n=3$) the problem is open.

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