19. *On the Theorem of Cauchy-Kowalevsky for First Order Linear Differential Equations with Degenerate Principal Symbols*

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Let

(1) $$P=\sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}+b(x)$$

be a first order linear differential operator with analytic coefficients defined at the origin of C^n. In this note, we discuss the following problem: Consider the differential equation

(2) $$Pu=f.$$

f and u being analytic functions at the origin, what condition should f satisfy for the existence of a local solution u of the equation (2) and how many solutions exist when f satisfies the condition? That is, our problem is to clarify the kernel and cokernel of the operator P. When $n=1$, Komatsu [2] and Malgrange [3] have a deep result for the index of the operator P, which is not necessarily of the first order.

Let $\mathcal{O}$ be the stalk at the origin of the sheaf of holomorphic functions over C^n. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the ideals of $\mathcal{O}$ generated by $a_1(x), \cdots, a_n(x)$ and $a_1(x), \cdots, a_n(x), b(x)$ respectively. In the case when $\mathfrak{A}$ is equal to $\mathcal{O}$, the answer to this problem is well-known as the theorem of Cauchy-Kowalevsky. In this note, therefore, we assume that $\mathfrak{A}$ is a proper ideal of $\mathcal{O}$. Such equations are used by Hadamard [1] to construct the elementary solution of a second order linear partial differential equation and by Sato-Kawai-Kashiwara [4] to determine the structure of pseudo-differential equations. We want to have general theory about the equation of such type. First we give the following conditions to formulate a theorem. We discuss examples which do not satisfy these conditions later.

(A) *$\mathfrak{A}$ is a proper and simple ideal of $\mathcal{O}$.*

Let $M=(\partial(a_1, \cdots, a_n)/\partial(x_1, \cdots, x_n))(0)$ be the Jacobian matrix of $a_1, \cdots, a_n$ at the origin. Let $M^*=J_1\oplus\cdots\oplus J_m\oplus J'_1\oplus\cdots\oplus J'_{m'}$ be the Jordan canonical matrix of M, where $J_i(1\leqslant i\leqslant m)$ and $J'_j(1\leqslant j\leqslant m')$ are the matrices of the Jordan blocks of sizes N_i and N'_j with eigenvalues $\lambda_i\neq 0$ and $\lambda'_j=0$ respectively.

(B) i) $N'_j=1\ (1\leqslant j\leqslant m')$.

ii) *There exists a real number* θ, *such that* $\theta<\arg\lambda_i<\theta+\pi$ *for* $1\leqslant i\leqslant m$, *where we denote by* $\arg\lambda_i$ *the argument of complex number* λ_i.

(C) *The equation* $b(0)=0$ *holds or* $b(0)+\sum_{i=1}^m l_i\lambda_i\neq 0$ *for arbitrary non-negative integers* $l_1,\cdots,l_m$.

Remark. (C) holds if condition (B) ii) holds, $b(0)\neq 0$ and $\theta<\arg b(0)<\theta+\pi$ for θ of (B) ii).

Theorem. *Assuming conditions* (A), (B) *and* (C), *we have the following conclusion.*

$$\operatorname{Coker} P\simeq \mathcal{O}/\mathfrak{B} \text{ and } \operatorname{Ker} P\simeq \begin{cases}\mathcal{O}/\mathfrak{B}, & \text{if } \mathfrak{A}=\mathfrak{B},\\ 0 & \text{if } \mathfrak{A}\neq\mathfrak{B}.\end{cases}$$

That is, an analytic solution u of (2) exists locally if and only if $f\in\mathfrak{B}$. If $\mathfrak{A}\neq\mathfrak{B}$, u is uniquely determined by f, and if $\mathfrak{A}=\mathfrak{B}$, there is a one-one correspondence between the solutions u and the Cauchy data $u|_V$, where V is the variety defined by $\mathfrak{B}$.

Proof. Taking account of conditions (B) ii) and (C), there exists a positive number ε which satisfies

$$|l_1\lambda_1+\cdots+l_m\lambda_m+b(0)|\geqslant(l_1+\cdots+l_m)\varepsilon$$

for any non-negative integers $l_1,\cdots,l_m$. Multiplying P by a constant number, we may assume from the beginning ε is equal to 2, i.e.,

$$\left|\sum_{i=1}^m l_i\lambda_i+b(0)\right|\geqslant 2\sum_{i=1}^m l_i. \tag{3}$$

Taking a different coordinate system, M is transformed into $G^{-1}M\,G$, where G is the Jacobian matrix of the coordinate transformation. Then, under a suitable coordinate system $x_1',\cdots,x_n'$, M is equal to M^* and $P=\sum_{i=1}^n c_i(x')\partial/\partial x_i'+b(x')$. Let $k=N_1+\cdots+N_m$, $k'=n-k$, K_i be equal to j if $N_1+\cdots+N_{j-1}<i\leqslant N_1+\cdots+N_j$ and δ_i be equal to 1 if there exists j such that $N_1+\cdots+N_{j-1}<i<N_1+\cdots+N_j$ and 0 otherwise. Considering condition (A) and (B) i), it is clear that $\mathfrak{A}$ is generated by $c_1(x'),\cdots,c_k(x')$. Now we define the following coordinate system $y_1,\cdots,y_k,z_1,\cdots,z_{k'}$:

$$\begin{cases} y_i=c_i(x')/\lambda_{K_i}-\delta_i y_{i+1} & \text{for } 1\leqslant i\leqslant k,\\ z_j=x'_{k+j} & \text{for } 1\leqslant j\leqslant k'.\end{cases}$$

Under this coordinate system,

$$P=\sum_{i=1}^k a_i(y,z)\frac{\partial}{\partial y_i}+\sum_{j=1}^{k'}a_j'(y,z)\frac{\partial}{\partial z_j}+b(y,z), \tag{4}$$

where we denote by y and z coordinates $y_1,\cdots,y_k$ and $z_1,\cdots,z_{k'}$ respectively, and M is equal to M^* because

$$\frac{\partial(y_1,\cdots,y_k,z_1,\cdots,z_{k'})}{\partial(x_1',\cdots,x_n')}(0)$$

is the identity matrix, and $\mathfrak{A}$ is generated by $y_1,\cdots,y_k$.

Case 1. $\mathfrak{A}=\mathfrak{B}$.

It is sufficient to show that when $f(0,z)\equiv 0$, there exists a unique solution u of (2) satisfying the initial condition $u(0,z)=v(z)$ for any v.

We define a semi-order on the set of pairs of multi-indices (α,β), where $\alpha=(\alpha_1,\cdots,\alpha_k)$, $\beta=(\beta_1,\cdots,\beta_{k'})$ and where α_i and β_j are non-negative integers, in the following way:

We define $(\alpha,\beta)<(\alpha',\beta')$ when and only when

$$|\alpha|<|\alpha'|,\qquad (|\alpha|=\alpha_1+\cdots+\alpha_k \text{ etc.}),$$

or

$$|\alpha|=|\alpha'|,\qquad |\beta|<|\beta'|,$$

or

$$|\alpha|=|\alpha'|,\qquad |\beta|=|\beta'|,\qquad \sum_{i=1}^{k} i\alpha_i<\sum_{i=1}^{k} i\alpha'_i.$$

Set $a_i(y,z)=\sum_{\alpha>0}a_{i\alpha}(z)y^\alpha=\sum_{\alpha>0,\beta\geqslant 0}a_{i\alpha\beta}z^\beta y^\alpha$ etc. Then easily we have the unique solution $u(y,z)=\sum_{\alpha\geqslant 0}u_\alpha(z)y^\alpha=\sum_{\alpha\geqslant 0,\beta\geqslant 0}u_{\alpha\beta}z^\beta y^\alpha$ of a formal power series under the initial condition $u_0(z)=v(z)$ in the following way. Let $\mathcal{J}$ be the ideal of the ring of formal power series generated by all $y^{\alpha'}z^{\beta'}$ which satisfy $(\alpha',\beta')>(\alpha,\beta)$. Then we have

$$P(u_{\alpha\beta}z^\beta y^\alpha)\equiv\sum_{i=1}^{k}\left(\lambda_{K_i}y_i\frac{\partial}{\partial y_i}+\delta_i y_{i+1}\frac{\partial}{\partial y_i}\right)u_{\alpha\beta}z^\beta y^\alpha \qquad \mathrm{mod}\ \mathcal{J}$$

$$\equiv u_{\alpha\beta}\sum_{i=1}^{k}\alpha_i\lambda_{K_i}z^\beta y^\alpha \qquad \mathrm{mod}\ \mathcal{J},$$

because $\mathfrak{A}$ is generated by $y_1,\cdots,y_k$, $M=M^*$ and $b(y,z)\in\mathfrak{A}$. Therefore, comparing the coefficients of $z^\beta y^\alpha$ of both sides of the equation (2), we can determine $u_{\alpha\beta}$ by (5) inductively:

$$(5)\qquad \begin{cases}(\sum_{i=1}^{k}\alpha_i\lambda_{K_i})u_{\alpha\beta}=\text{a number determined only by } u_{\alpha'\beta'} \text{ which satisfy} \\ \qquad\qquad \text{the relation } (\alpha',\beta')<(\alpha,\beta).\end{cases}$$

Then we can prove by the method of majornant that u is analytic at the origin. In fact, for suitable positive numbers r, C and C' we have

$$(6)\qquad \begin{cases} a_i(y,z)-\lambda_{K_i}y_i-\delta_i y_{i+1}\ll\dfrac{Cs(s+t)}{r-(s+t)} & \text{for } 1\leqslant i\leqslant k, \\ a'_j(y,z)\ll\dfrac{Cs(s+t)}{r-(s+t)} & \text{for } 1\leqslant j\leqslant k', \\ b(y,z)\ll\dfrac{Cs}{r-(s+t)},\quad v(z)\ll\dfrac{C'}{r-t},\quad f(y,z)\ll\dfrac{C's}{r-(s+t)}, \end{cases}$$

where we define $s=y_1+\cdots+y_k$, $t=z_1+\cdots+z_{k'}$. Taking account of (3), (5) and (6), we have easily the relation $\varphi\gg u$ if a formal power series φ satisfies

$$(7)\qquad \begin{cases} P^*\varphi\gg\dfrac{C's}{r-(s+t)} \quad\text{and}\quad \varphi(0,z)\gg\dfrac{C'}{r-t}, \\ \text{where } P^*=\displaystyle\sum_{i=1}^{k}\left(2y_i-\frac{Cs(s+t)}{r-(s+t)}\right)\frac{\partial}{\partial y_i}-\sum_{i=1}^{k-1}y_{i+1}\frac{\partial}{\partial y_i} \\ \qquad\qquad -\dfrac{Cs(s+t)}{r-(s+t)}\displaystyle\sum_{i=1}^{k'}\frac{\partial}{\partial z_j}-\frac{Cs}{r-(s+t)}. \end{cases}$$

On the other hand, the solution φ of

$$(8)\quad \begin{cases} \left(1-k\dfrac{C(s+t)}{r-(s+t)}\right)\dfrac{\partial\varphi}{\partial s}-k'\dfrac{C(s+t)}{r-(s+t)}\dfrac{\partial\varphi}{\partial t}-\dfrac{C}{r-(s+t)}\varphi \\ \quad =\dfrac{C'}{r-(s+t)}, \qquad \varphi(0,t)=\dfrac{C'}{r-t} \end{cases}$$

is analytic at the origin, which is clear by the theorem of Cauchy-Kowalevsky, so we come to the conclusion, because φ satisfies (7). In fact,

$$P^*\varphi=y_1\frac{\partial\varphi}{\partial s}+\frac{C's}{r-(s+t)}\gg\frac{C's}{r-(s+t)} \quad \text{and} \quad \varphi(0,z)=\frac{C'}{r-t}.$$

Case 2. $\mathfrak{A}\neq\mathfrak{B}$.

It is sufficient to show that there exists a unique solution u of (2) when f belongs to $\mathfrak{B}$.

First we have by (5)′ the unique solution of a formal power series as in Case 1:

$$(5)'\quad \begin{cases} u_0(z)=f_0(z)/b_0(z), \text{ which is analytic because } f\in\mathfrak{B}, \\ (\sum_{i=1}^{k}\alpha_i\lambda_{K_i}+b(0))u_{\alpha\beta}=\text{a number determined only by } u_{\alpha'\beta'} \text{ which} \\ \quad \text{satisfy the relation } (\alpha',\beta')<(\alpha,\beta), \text{ where we use the same} \\ \quad \text{notations as in Case 1.} \end{cases}$$

We have the following majorant series as in Case 1:

$$(6)'\quad \begin{cases} f(0,z)/b(0,z)\ll\dfrac{C'}{r-t}, \qquad f(y,z)-f(0,z)\ll\dfrac{C's}{r-(s+t)}, \\ b(y,z)-b(0,0)\ll\dfrac{C(s+t)}{r-(s+t)} \text{ and the others are the same as in} \\ \quad \text{Case 1.} \end{cases}$$

As in Case 1, we can prove the existence of φ which is analytic at the origin and satisfies

$$P^*\varphi\gg\frac{C's}{r-(s+t)} \quad \text{and} \quad \varphi(0,z)\gg\frac{C'}{r-t},$$

$$P^*=\sum_{i=1}^{k}\left(2y_i-\frac{2Cs(s+t)}{r-(s+t)}\right)\frac{\partial}{\partial y_i}-\sum_{i=1}^{k-1}y_{i+1}\frac{\partial}{\partial y_i}-\frac{Cs(s+t)}{r-(s+t)}\sum_{j=1}^{k'}\frac{\partial}{\partial z_j}.$$

Considering (3), (5)′, (6)′, (7)′ and $z^\beta y^\alpha\ll s\sum_{i=1}^{k}(\partial/\partial y_i)z^\beta y^\alpha$ for $|\alpha|>0$, we see that φ is a majorant series of u, so u is analytic. This completes the proof of the theorem.

We give some examples which do not satisfy (A), (B) or (C).

1) $P=x_1\dfrac{\partial}{\partial x_1}+x_2^2\dfrac{\partial}{\partial x_2}$, Ker $P\simeq \boldsymbol{C}$, Im $P\not\ni x_1x_2$.

2) $P=x_1\dfrac{\partial}{\partial x_1}+x_2^2\dfrac{\partial}{\partial x_2}+1$, Ker $P=0$, Im $P\not\ni x_1x_2, x_2$.

3) $P=x_2\dfrac{\partial}{\partial x_1}+1$, Ker $P=0$, Im $P\not\ni (1-x_1)^{-1}$.

4) $P=x_2\dfrac{\partial}{\partial x_1}+x_3\dfrac{\partial}{\partial x_2}$, Ker $P\ni x_2^2-2x_1x_3$, Im $P\not\ni x_2^2$.

5) $P=x_2\dfrac{\partial}{\partial x_1}+x_4\dfrac{\partial}{\partial x_3}$, Ker $P \ni x_1x_4-x_2x_3$, Im $P \not\ni x_1x_4$.

6) $P=x_2\dfrac{\partial}{\partial x_1}+x_3\dfrac{\partial}{\partial x_3}$, Ker $P=\{f(x_2)\,;\,f\in\mathcal{O}_1\}$,

Im $P \not\ni x_3(1-x_1)^{-1}$, where we denote by $\mathcal{O}_1$ the stalk at the origin of the sheaf of holomorphic functions over $\boldsymbol{C}^1$.

7) $P'=P+1$, where P is the same as in 4), 5) or 6,

Ker $P'=0$, Im $P' \not\ni (1-x_1)^{-1}$.

8) $P=x_1\dfrac{\partial}{\partial x_1}-x_2\dfrac{\partial}{\partial x_2}$, Ker $P\simeq$ Coker $P\simeq\{f(x_1x_2)\,;\,f\in\mathcal{O}_1\}$.

9) $P=x_1\dfrac{\partial}{\partial x_1}-\lambda x_2\dfrac{\partial}{\partial x_2}$, where λ is a positive irrational number, Ker $P\simeq\boldsymbol{C}$. If $f(0)=0$, the equation $Pu=f$ has a solution of a formal power series, but it is a problem of Diophantine approximation whether the series converges or not. Let a_n, b_n and λ be numbers satisfying $a_1=1$, $a_{n+1}\geqslant 2a_n\,!$, $\lambda=\sum_{n=1}^{\infty}1/a_n$ and $b_n<a_n\lambda<b_n+1$, where a_n and b_n are integers, and f be equal to $1-(1-x_1-x_2)^{-1}$. Then the formal solution is not analytic because its coefficient of $x_1^{b_n}x_2^{a_n}$ is larger than $a_n\,!$. On the other hand, when λ is an algebraic number, we see that the formal solution is always analytic at the origin by the theorem of Roth.

10) $P=x_1\dfrac{\partial}{\partial x_1}+x_2\dfrac{\partial}{\partial x_2}-1$, Ker $P\simeq$ Coker $P\simeq\{Cx_1+C'x_2\,;\,C, C'\in\boldsymbol{C}\}$.

Remark. In the case 1), 2), 3), 6) and 7), a similar result holds as in the theorem if we think P in the category of formal power series, for instance, in 3), $u=\sum_{i,j\geqslant 0}(-1)^j((i+j)\,!/i\,!)x_1^ix_2^j$ satisfies $Pu=(1-x_1)^{-1}$.

We give finally the following examples satisfying (A), (B) and (C).

11) $P=(x_1+x_2)\dfrac{\partial}{\partial x_1}+(x_2+x_3x_4)\dfrac{\partial}{\partial x_2}+2x_3\dfrac{\partial}{\partial x_3}+x_2\dfrac{\partial}{\partial x_4}$,

Ker $P\simeq$ Coker $P\simeq\{f(x_4)\,;\,f\in\mathcal{O}_1\}$,

$P'=P-3/2$, Ker $P'=$ Coker $P'=0$,

$P''=P+x_3+x_4^2$, Ker $P''=0$, Coker $P''\simeq\{C+C'x_4\,;\,C, C'\in\boldsymbol{C}\}$.

References

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