# 38. *A von Neumann Algebra Continuous over a von Neumann Subalgebra*

By Marie CHODA
Department of Mathematics, Osaka Kyoiku University

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1973)

**1.** In [3], we have introduced a generalized notion of abelian projections of von Neumann algebras and proved that some elementary properties of abelian projections are preserved under the generalization.

Using this concept, in this note, we shall introduce a notion that a von Neumann algebra is continuous over a von Neumann subalgebra, and show some properties of such a von Neumann algebra in § 2.

In § 3, we shall prove that a von Neumann algebra $\mathcal{A}$ continuous over a von Neumann subalgebra $\mathcal{B}$ has an useful property relative to an expectation of $\mathcal{A}$ onto $\mathcal{B}$. In [2], H. Choda has introduced the notion of Maharam subalgebras of von Neumann algebras motivated by Maharam's lemma. On the analogy of this definition, we shall introduce a notion of strong Maharam subalgebras of von Neumann algebras and prove that a von Neumann subalgebra $\mathcal{B}$ of a von Neumann algebra $\mathcal{A}$ contained in the center is a strong Maharam subalgebra of $\mathcal{A}$ if $\mathcal{A}$ is continuous over $\mathcal{B}$.

We shall use the terminology due to Dixmier [4] throughout this note without further explanations.

**2.** In the sequel, let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$. Denote by $\mathcal{B}^c=\mathcal{A}\cap\mathcal{B}'$ the relative commutant of $\mathcal{B}$ in $\mathcal{A}$ and $\mathcal{B}^P$ the set of all projections in $\mathcal{B}$.

The following definition is introduced in [3] as a generalization of the notion of abelian projections:

**Definition 1.** A projection $E\in\mathcal{A}$ is called to be *abelian over* $\mathcal{B}$ if $E\in\mathcal{B}^c$ and, for every projection $P\in\mathcal{A}$ such that $P\leqq E$, there exists a projection $Q\in\mathcal{B}$ such that $P=QE$.

The following lemma gives an alternative algebraic definition of abelian projections over $\mathcal{B}$:

**Lemma 2.** *$E\in\mathcal{A}^P$ is abelian over $\mathcal{B}$ if and only if $E\in\mathcal{B}^c$ and $E\mathcal{A}E=\mathcal{B}E$.*

**Proof.** The "only if" part is obvious. Conversely, let $E$ be a projection in $\mathcal{B}^c$ such that $E\mathcal{A}E=\mathcal{B}E$ and $\tilde{E}$ the $\mathcal{B}\cap\mathcal{B}'$-support of $E$, that is,

$$\tilde{E}=\inf\{F\in(\mathcal{B}\cap\mathcal{B}')^P\,;\,F\geqq E\},$$

then the induction of $\mathcal{B}_{\tilde{E}}$ to $\mathcal{B}_E$ is an isomorphism [4; p. 19, Proposition 2]. Denote by $R|_F$ the restriction of $R$ onto $F\mathfrak{H}$ for $R \in \mathcal{A}^P$ and $F \in \mathcal{A}^P$, where $\mathfrak{H}$ is the Hilbert space on which $\mathcal{A}$ acts. Take $P \in (E\mathcal{A}E)^P = (E\mathcal{B}E)^P$, then there exists $Q \in (\tilde{E}\mathcal{B}\tilde{E})^P$ such that $(Q|_{\tilde{E}})_E = P|_E$. Therefore, we have $Q \in \mathcal{B}^P$ satisfying $P|_E = EQE|_E$, which implies $P = QE$.

**Definition 3.** A von Neumann algebra $\mathcal{A}$ is called to be *continuous over* $\mathcal{B}$ if $\mathcal{A}$ contains non-zero projections abelian over $\mathcal{B}$.

**Remark.** If $\mathcal{A}$ is abelian and continuous over $\mathcal{B}$, then $\mathcal{B}$ is called by Dye [5] a type II subalgebra. For the center $\mathcal{Z}$ of $\mathcal{A}$, $\mathcal{A}$ is continuous over $\mathcal{Z}$ if and only if $\mathcal{A}$ is continuous in the usual sense, cf. [4]. If $\mathcal{B}$ is contained in the center $\mathcal{Z}$ of $\mathcal{A}$ then this definition is introduced by Guichardet, cf. [3] and [7].

**Example 4.** A continuous von Neumann algebra $\mathcal{A}$ is continuous over an abelian subalgebra $\mathcal{B}$. In fact, if $\mathcal{A}$ is not continuous over $\mathcal{B}$, then by Lemma 2, there exists a non-zero projection $E \in \mathcal{A} \cap \mathcal{B}'$ with $\mathcal{A}_E = \mathcal{B}_E$. Hence $\mathcal{A}$ contains a non-zero abelian projection.

**Example 5.** A maximal abelian subalgebra $\mathcal{B}$ of a continuous von Neumann algebra $\mathcal{A}$ is continuous over $\mathcal{Z}$, cf. Lemma 13.

**Proposition 6.** *If $\mathcal{A}$ is continuous over $\mathcal{B}$, then $\mathcal{B}'$ is continuous over $\mathcal{A}'$.*

**Proof.** If $\mathcal{B}'$ is not continuous over $\mathcal{A}'$, then we have a non-zero projection $E \in \mathcal{A} \cap \mathcal{B}'$ with $\mathcal{B}'_E = \mathcal{A}'_E$; hence we have $\mathcal{A}_E = \mathcal{B}_E$ and a contradiction.

**Theorem 7.** *If every projection in $\mathcal{B}^c$ is decomposed into the sum of two projections in $\mathcal{B}^c$ which are orthogonal and equivalent* (mod. $\mathcal{B}^c$), *then $\mathcal{A}$ is continuous over $\mathcal{B}$.*

**Proof.** Assume that $\mathcal{A}$ is not continuous over $\mathcal{B}$. Let $E$ be a non-zero projection abelian over $\mathcal{B}$ and $E = Q + R$ a decomposition in the assumption. Let $\bar{Q}$ (resp. $\bar{R}$) be the $\mathcal{B}$-support of $Q$ (resp. $R$). Then $Q \sim R$ (mod. $\mathcal{B}^c$) implies $\bar{Q} = \bar{R}$. In fact, if $P \in \mathcal{B}$ is a projection and $V \in \mathcal{B}^c$ a partial isometry with $VQV^* = R$; then, $PR = R$ if and only if $VPQV^* = VQV^*$ if and only if $PQ = Q$, which implies $\bar{Q} = \bar{R}$. Since $E$ is abelian over $\mathcal{B}$, $Q = \bar{Q}E$ and $R = \bar{R}E$ by [3; Lemma 2]. Hence $Q = R$ or $E = 0$, which is a contradiction.

**3.** A positive linear mapping $e$ of $\mathcal{A}$ onto $\mathcal{B}$ is called an *expectation* of $\mathcal{A}$ onto $\mathcal{B}$ if $e$ satisfies the following conditions: (i) $I^e = I$ and (ii) $(BAC)^e = BA^eC$ for all $A \in \mathcal{A}$ and for all $B, C \in \mathcal{B}$, cf. [8] and [9].

**Definition 8.** Let $e$ be a normal expectation of $\mathcal{A}$ onto $\mathcal{B}$. $\mathcal{B}$ is called an *e-strong Maharam subalgebra* of $\mathcal{A}$ if for any $P \in \mathcal{A}^P$ and any $B \in \mathcal{B}$ such that $0 \leqq B \leqq P^e$ there exists a projection $Q$ such that $Q \leqq P$ and $Q^e = B$, cf. [2].

**Lemma 9.** *If $\mathcal{A}$ is continuous over $\mathcal{B}$, then, for every $P \in \mathcal{A}^P$ and $Q \in \mathcal{B}^P$ satisfying $0 \neq PQ \in (\mathcal{B}^c)^P$, there exist two projections $R \in \mathcal{A}^P$ and $E \in \mathcal{B}^P$ such that $0 \neq R \leqq PQ$, $0 \neq E \leqq Q$ and $(P-R)F \neq 0$ and $RF \neq 0$ for any $F \in \mathcal{B}^P$ with $0 \neq F \leqq E$.*

**Proof.** Assume that there are not such projections. Take any $R \in \mathcal{A}^P$ such as $0 \neq R \leqq PQ$. Put

$$G = \sup \{G \in \mathcal{B}^P ; (P-R)G = 0 \text{ and } G \leqq Q\}$$

and

$$G' = \sup \{G' \in \mathcal{B}^P ; RG' = 0 \text{ and } G' \leqq Q-G\}.$$

Then $G \in \mathcal{B}$ and $G' \in \mathcal{B}$. If $G' \neq Q-G$, then $0 \neq Q-G-G' \leqq Q$. By the assumption, for $R$ and $Q-G-G'$, there exists a nonzero projection $F \in \mathcal{B}$ such that $F \leqq Q-G-G'$ and $(P-R)F=0$ or $RF=0$. By the definitions of $G$ and $G'$, if $(P-R)F=0$ or $RF=0$, then $F=0$, which is a contradiction. Therefore $G'=Q-G$, which implies $R(Q-G)=0$, so that $R=RQ=RG=PG=PQG=GPQ$. Therefore $PQ$ is abelian over $\mathcal{B}$, which contradicts that $\mathcal{A}$ is continuous over $\mathcal{B}$.

Let $e$ be a normal expectation of $\mathcal{A}$ onto $\mathcal{B}$, then the restriction $e|_{\mathcal{B}^c}$ is a normal expectation of $\mathcal{B}^c$ to $\mathcal{B}$ if $\mathcal{B}$ is abelian, which will be identified with $e$.

**Theorem 10.** *Let $\mathcal{B}$ be an abelian subalgebra of $\mathcal{A}$ and $e$ a normal expectation of $\mathcal{A}$ onto $\mathcal{B}$. If $\mathcal{B}^c$ is continuous over $\mathcal{B}$, then $\mathcal{B}$ is an $e$-strong Maharam subalgebra in $\mathcal{B}^c$.*

**Proof.** At first, we shall show

(*) Take and fix a projection $P$ of $\mathcal{B}^c$. For any integer $n$ and projection $R \in \mathcal{B}$ with $PR \neq 0$, there exists a projection $E \in \mathcal{B}^c$ such that $0 \neq E \leqq P$, $0 \neq ER$ and $E^e R \leqq P^e R / 2^n$.

By Lemma 9, there exist two projections $G$ in $\mathcal{B}^c$ and $E$ in $\mathcal{B}$ satisfying (i) $0 \neq G \leqq PR$, (ii) $0 \neq E \leqq R$ and (iii) $(P-G)F \neq 0$ and $GF \neq 0$ for any projection $F \in \mathcal{B}$ with $0 \neq F \leqq E$. Since $\mathcal{B}$ is abelian, then $\mathcal{B}$ is isomorphic to $C(\Omega)$, the algebra of all continuous functions on the character space $\Omega$ of $\mathcal{B}$. Denote by $C$ (resp. $D$) the projection corresponding to the characteristic function of the set $\{\omega \in \Omega ; 2G^e(\omega) \geqq (\text{resp. } \leqq) P^e(\omega)\}$. Put $Q=(P-G)C+GD$, then $Q^e=(P^e-G^e)C+G^eD \leqq P^e/2$. Since $CE \neq 0$ or $DE \neq 0$ by the definitions, we have $(P-G)CE \neq 0$ or $GDE \neq 0$. Hence we have $QE=(P-G)CE+GDE \neq 0$. Therefore, for $R$ in (*) and $n=1$, we have a projection $Q$ in $\mathcal{B}^c$ such that $Q \leqq P$, $QR \neq 0$ and $Q^eR \leqq P^eR/2$. Replacing $P$ by $Q$ and repeating the process, we have (*).

Next, using (*), we shall show the following:

(**) For any nonzero projection $P \in \mathcal{B}^c$ and $B \in \mathcal{B}$ with $0 < B \leqq P^e$, there exists a nonzero projection $Q \in \mathcal{B}^c$ such that $Q \leqq P$ and $Q^e \leqq B$.

By the spectral theorem, there exist a nonzero projection $R \in \mathcal{B}$ and an integer $n$ such that $BR \geqq R/2^n$. Since $P^e \geqq B$ by the hypothesis, $(PR)^e = P^e R \geqq BR \geqq R/2^n > 0$, which implies $PR \neq 0$. By (*), there is a nonzero projection $E \in \mathcal{B}^c$ such that $E \leqq P$, $ER \neq 0$ and $E^e R \leqq P^e R/2^n$. On the other hand $B \leqq P^e \leqq 1$, so that we have

$$(ER)^e = E^e R \leqq P^e R/2^n \leqq R/2^n \leqq BR \leqq B.$$

Put $Q = ER$. Then $Q$ is a nonzero projection belonging to $\mathcal{B}^c$ with $Q \leqq P$ and $Q^e \leqq B$. Therefore (**) is proved.

Let $P$ be a projection in $\mathcal{B}^c$ and $B \in \mathcal{B}$ such as $0 \leqq B \leqq P^e$. By (**) and Zorn's lemma, there exists a maximal orthogonal family $(Q_\alpha)_{\alpha \in I}$ of projections in $\mathcal{B}^c$ such that $0 \neq Q_\alpha \leqq P$ for each $\alpha \in I$ and $\sum_\alpha Q_\alpha^e \leqq B$. Put $Q = \sum_\alpha Q_\alpha$, then $Q$ is a projection of $\mathcal{B}^c$ and $Q \leqq P$. By the normality of $e$, $Q^e = \sum_\alpha Q_\alpha^e$. If $Q^e \neq B$, then we have $(P-Q)^e \geqq B - Q^e > 0$, so that, by (**), there exists a projection $R \in \mathcal{B}^c$ such that $0 < R \leqq P - Q$ and $R^e \leqq B - Q^e$, which contradicts the maximality of $(Q_\alpha)_{\alpha \in I}$. Hence $Q^e = B$. Thus, for any projection $P \in \mathcal{B}^c$ and $B \in \mathcal{B}$ such as $0 \leqq B \leqq P^e$, we have a projection $Q \in \mathcal{B}^c$ such that $Q \leqq P$ and $Q^e = B$. This completes the proof.

Especially, let $\mathcal{A}$ be an abelian von Neumann algebra, then Theorem 9 contains Maharam's lemma, cf. [5]. Our proofs of Theorem 10 and Lemma 9 are analogous to H. Choda's proof [1] of Maharam's lemma based on Dye's sketch [5].

**Corollary 11.** *Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$ contained in the center of $\mathcal{A}$ and $e$ a normal expectation of $\mathcal{A}$ onto $\mathcal{B}$. If $\mathcal{A}$ is continuous over $\mathcal{B}$, then $\mathcal{B}$ is an e-strong Maharam subalgebra in $\mathcal{A}$.*

Let $\mathcal{A}$ be a finite von Neumann algebra, then there exists the natural mapping (center-valued Dixmier's trace) $\natural$ of $\mathcal{A}$ onto the center $\mathcal{Z}$, which is a faithful normal (conditional in the sense of [9]) expectation of $\mathcal{A}$ onto $\mathcal{Z}$. If $\mathcal{A}$ is continuous, then $\mathcal{A}$ is continuous over $\mathcal{Z}$. Hence, as a consequence of Corollary 11, we have the following

**Corollary 12.** *Let $\mathcal{A}$ be a finite continuous von Neumann algebra and $\natural$ be the natural mapping of $\mathcal{A}$ onto the center $\mathcal{Z}$, then $\mathcal{Z}$ is a $\natural$-strong Maharam subalgebra of $\mathcal{A}$.*

Now, we shall discuss another application of Theorem 10.

**Lemma 13.** *Let $\mathcal{B}$ be a maximal abelian subalgebra of $\mathcal{A}$ and $\mathcal{C}$ a von Neumann subalgebra of $\mathcal{A}$ contained in the center $\mathcal{Z}$. If $\mathcal{A}$ is continuous over $\mathcal{C}$, then $\mathcal{B}$ is continuous over $\mathcal{C}$.*

**Proof.** If $\mathcal{B}$ is not continuous over $\mathcal{C}$, then there is a nonzero projection $E \in \mathcal{B}$ which is abelian over $\mathcal{C}$, that is, $\mathcal{B}_E = \mathcal{C}_E$ by Lemma 2. Since $\mathcal{B}$ is maximally abelian in $\mathcal{A}$, $\mathcal{B}_E$ is maximally abelian in $\mathcal{A}_E$. Hence

$$\mathcal{C}_E=\mathcal{B}_E=\mathcal{B}'_E\cap\mathcal{A}_E=\mathcal{C}'_E\cap\mathcal{A}_E=\mathcal{A}_E.$$

Therefore, by Lemma 2, $E$ is abelian over $\mathcal{C}$ in $\mathcal{A}$, which contradicts that $\mathcal{A}$ is continuous over $\mathcal{C}$.

**Corollary 14.** *Let $\mathcal{B}$ be a maximal abelian subalgebra of $\mathcal{A}$ and $\mathcal{C}$ a von Neumann subalgebra of $\mathcal{A}$ contained in the center. If $\mathcal{A}$ is continuous over $\mathcal{B}$, then $\mathcal{B}$ is continuous over $\mathcal{C}$.*

**Proof.** If $\mathcal{B}$ is not continuous over $\mathcal{C}$, then by the proof of Lemma 13 there exists a nonzero projection $E\in\mathcal{B}$ such that $\mathcal{B}_E=\mathcal{A}_E$. Since $\mathcal{B}$ is abelian, $E\in\mathcal{B}^c$. Therefore, by Lemma 2, $E$ is abelian over $\mathcal{B}$, which contradicts that $\mathcal{A}$ is continuous over $\mathcal{B}$.

**Theorem 15.** *Let $\mathcal{A}$ be a finite continuous von Neumann algebra and $\natural$ the natural mapping of $\mathcal{A}$ onto the center $\mathcal{Z}$. Then $\mathcal{Z}$ is a $\natural$-strong Maharam subalgebra of each maximal abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$.*

**Proof.** For each maximal abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$, $\mathcal{Z}$ is contained in $\mathcal{B}$. In fact, $\mathcal{Z}\subset\mathcal{A}\cap\mathcal{B}'=\mathcal{B}$. Since $\mathcal{A}$ is continuous, $\mathcal{B}$ is continuous over $\mathcal{Z}$ by Lemma 13. Therefore, by Theorem 10, $\mathcal{Z}$ is a $\natural$-strong Maharam subalgebra of $\mathcal{B}$.

**Remark.** Theorem 15 (resp. Corollary 12) is a sharpening of the theorem of Feldman [6] (resp. [4; p. 218]).

## References

[1] H. Choda: On a Maharam's lemma (in Japanese). Reports of Sakkokai Symposium on Functional Analysis at Hachioji, 97–105 (1967).

[2] ——: On Maharam subfactors of finite factors. Proc. Japan Acad., **43**, 451–455 (1967).

[3] M. Choda: Abelian projections over a von Neumann subalgebra. Proc. Japan Acad., **48**, 384–388 (1972).

[4] J. Dixmier: Les algèbres d'opérateurs dans l'espace Hilbertien. Gauthier-Villars, Paris (1957).

[5] H. A. Dye: On groups of measure preserving transformations. I. Amer. J. Math., **81**, 119–159 (1959).

[6] J. Feldmann: Isomorphisms of finite type II rings of operators. Ann. of Math., **63**, 565–571 (1956).

[7] A. Guichardet: Une caractérisation des algèbres de von Neumann discrètes. Bull. Soc. Math. France, **89**, 77–101 (1961).

[8] M. Nakamura and T. Turumaru: Expectations in an operator algebra. Tohoku Math. J., **6**, 182–188 (1954).

[9] H. Umegaki: Conditional expectation in an operator algebra. Tohoku Math. J., **6**, 177–181 (1954).