36. On the Boundary Value Problem for Elliptic System of Linear Differential Equations. II

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In this note we give the sketch of the proof of Theorem 1 in our preceding note [4] after giving another application (Example 4) of our theorem.

In this note we will use the same notations as in our previous note [4] and will not repeat their definition.

Further details of this note will appear elsewhere.

Example 4. In Example 2 of our previous note we treated only the Dirichlet problem. However Theorem 1, together with the results in Sato-Kawai-Kashiwara [5], [6], makes it possible to treat the so-called "non-elliptic boundary value problems" (cf. Hörmander [3], Egorov-Kondratev [1] and Sjöstrand [7]).

Let M be a real analytic manifold and N be a submanifold of M. For the sake of simplicity, we assume that N is of codimension 1 and defined by $\varphi(x)=0$. As in Example 2, $M=M_+\cup M_-\cup N, S_N^*M=N_+\cup N_-$. Then S_N^*X decomposes into three parts, namely, $S_\pm=q^{-1}(N_\pm)$ and $\sqrt{-1}S^*M$. Let $\mathcal{M}$ be an elliptic system of differential equations. For the sake of simplicity, we assume that $\mathcal{E}xt^j_{\mathcal{D}_M}(\mathcal{M},\mathcal{A}_M)=0$ for $j>0$. We denote by $\mathcal{S}$ the solution sheaf $\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M},\mathcal{A}_M)$. We set $\mathcal{M}_\pm =p_*(\mathcal{P}_{Y\to X}\otimes_{\mathcal{D}_X}\mathcal{M}|_{S_\pm})$. Then $\mathcal{M}_\pm$ is a system of pseudo-differential equations on $\sqrt{-1}S^*N$, so that $\mathcal{M}_N=p_*(\mathcal{P}_{Y\to X}\otimes_{\mathcal{D}_X}\mathcal{M})$ is a direct sum of $\mathcal{M}_+$ and $\mathcal{M}_-$. Suppose that an admissible $\mathcal{P}_N$-subModule $\mathcal{N}$ of $\mathcal{M}_-$ is given so that the quotient sheaf $\mathcal{L}=\mathcal{M}_-/\mathcal{N}$ is also admissible. By Theorem 1, we have

$$(j_{+*}(\mathcal{S})/\mathcal{S})|_N=\pi_{N*}\mathcal{H}om_{\mathcal{P}_N}(\mathcal{M}_-,\mathcal{C}_N).$$

Hence we obtain an exact sequence

$$0\longrightarrow\pi_{N*}\mathcal{H}om_{\mathcal{P}_N}(\mathcal{L},\mathcal{C}_N)\longrightarrow(j_{+*}(\mathcal{S})/\mathcal{S})|_N \xrightarrow{B}\pi_{N*}\mathcal{H}om_{\mathcal{P}_N}(\mathcal{N},\mathcal{C}_N)\xrightarrow{\delta}\pi_{N*}\mathcal{E}xt^1_{\mathcal{P}_N}(\mathcal{L},\mathcal{C}_N). \tag{1}$$

The generalized boundary value problem means the problem to find a solution u of $\mathcal{M}$ defined on M_+ satisfying a boundary condition $Bu=\mu$, where μ is a given microfunction solution of $\mathcal{N}$. Therefore $\mathcal{E}xt^1_{\mathcal{P}_N}(\mathcal{L},\mathcal{C}_N)=0$ implies the existence of u for every μ and $\mathcal{H}om_{\mathcal{P}_N}(\mathcal{L},\mathcal{C}_N)=0$ implies the uniqueness of u (modulo the solutions defined on a neighborhood of N).

In the case when $\mathcal{E}xt^1_{\mathcal{P}_N}(\mathcal{L}, C_N)$ does not vanish, u exists if and only if $\delta\mu=0$. The structure of the sheaf $\mathcal{E}xt^1_{\mathcal{P}_N}(\mathcal{L}, C_N)$ is investigated so well that we can write down the compatibility condition $\delta\mu=0$ in many cases (Corollary 2.4.2 of Chapter III of $S-K-K$ [5] and Theorem 1 of Sato-Kawai-Kashiwara [6]).

We will illustrate the above discussion concretely when $\mathcal{M}$ is a single equation. Let $P(x, D)$ be an elliptic differential operator of order $m=2r$ and $\mathcal{M}$ be a system $\mathcal{D}_M/\mathcal{D}_M P$. If we denote the generator of $\mathcal{M}$ by u, then $\mathcal{M}=\mathcal{D}_M u$ with the fundamental relation $P(x, D)u=0$. We take a local coordinate system $(x_1, \cdots, x_n)$ of M so that $\varphi(x)=x_1$. Then the preparation theorem for pseudo-differential operators (Lemma 3.5.2 of $S-K-K$ [5] Chapter II) asserts that $P(x, D)$ has the form $Q^+(x, D)Q^-(x, D)$, where $Q^-(x, D)=D_1^r-\sum_{j=0}^{r-1} Q_{r-j}(x, D')D_1^j$ and the principal symbol of $Q^+(x, D)$ never vanishes on S_-. Here $D'=(D_2, \cdots, D_n)$.

Let $B_j(x, D)$ $(j=1, \cdots, r)$ be the pseudo-differential operators of order m_j of the form $\sum_{k=0}^{m_j} B_j^k(x', D')D_1^k$ where $B_j^k(x', D')$ is a pseudo-differential operator defined on $\sqrt{-1}S^*N$ of order at most m_j-k. We may assume without loss of generality that $m_j<m$. Then we consider the generalized boundary value problem for $\mathcal{M}=\mathcal{D}_M/\mathcal{D}_M P$ on $\{x_1\geqq 0\}$ by requiring that the solution u on $\{x_1>0\}$ of the equation $\mathcal{M}$ satisfy the relation $B_j(x, D)u|_{x_1=+0}=\mu_j(x')$ $(j=1, \cdots, r)$ for given $\mu_j(x')\in\mathcal{B}_N/\mathcal{A}_N$. In this case, $\mathcal{M}_-$ is a free $\mathcal{P}_N$-Module generated by $u_j(=D_1^j u|_N=1_{Y\to X}\otimes D_1^j u)$ for $j=0, \cdots, r-1$. (See $S-K-K$ [5].) We take as $\mathcal{N}$ the free $\mathcal{P}_N$-Module with the system of generations $\mu_1, \cdots, \mu_r$, and define the homomorphism

$$\iota: \mathcal{N}\to\mathcal{M}_- \quad \text{by} \quad \mu_j\mapsto B_j(x, D)u|_N=\sum_{k=0}^{m_j} B_j^k(x', D')u_k.$$

Then this is included in the framework which we described before, provided that ι is injective.

Theorem 1 in Kashiwara-Kawai [4] asserts that the local solvability of the above boundary value problem is equivalent to the solvability of the system of equations $(D_1^k Q^-(x, D)u)|_{x_1=+0}=0$ $(k=0, \cdots, r-1)$ and $B_j(x, D)u|_{x_1=+0}=\mu_j(x')$ $(j=1, \cdots, r)$, where the unknown functions are $D_1^j u(x)|_{x_1=+0}$ $(j=0, \cdots, m-1)$. Hence the local solvability of the above boundary value problem is reduced to the solvability of the following determined system of pseudo-differential equations (1) on $\sqrt{-1}S^*N$

(2)

$$\left(\begin{array}{lll} Q_r(0, x', D') & Q_{r-1}(0, x', D') & Q_{r-2}(0, x', D') \\ R_{r,0}(x', D') & Q_r(0, x', D')+R_{r,1}(x', D') & Q_{r-1}(0, x', D')+R_{r,2}(x', D') \\ \vdots & & \\ R_{r+j-1,0}(x', D') & R_{r+j-1,1}(x', D') & \\ & & B_j^i(x', D') \end{array}\right.$$

$$\cdots Q_1(0,x',D') \quad \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & \ddots & \\ & & 1 \end{pmatrix} \times \begin{pmatrix} u_0(x') \\ \vdots \\ \vdots \\ u_{m-1}(x') \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_1(x') \\ \vdots \\ \mu_r(x') \end{pmatrix}$$

where $R_{l,k}(x',D')$ are pseudo-differential operators of order at most $l-k$. Such an observation is a concrete form of the reasoning we developed using the extension group of the quotient $\mathcal{P}_N$-Module $\mathcal{L}=\mathcal{M}_-/\mathcal{N}$. In fact $\mathcal{L}$ is a system generated by $u_0, \cdots, u_{m-1}$ with the fundamental relation which is obtained from (2) setting $\mu_j=0$.

In passing the principal symbol of the matrix operator in the left hand side of (2) is

$$(3) \qquad \Delta(x',\eta')=\det\begin{pmatrix} q_r & q_{r-1} & \cdots q_1 & 1 & & 0 \\ 0 & q_r & \cdots & q_1 & 1 & \\ \vdots & & \ddots & & \ddots & \ddots \\ 0 & & q_r & & q_1 & 1 \\ & & & b_j^i & & \end{pmatrix}$$

$$=\det\Big(b_j^i+\sum_{\substack{i=k-l+1\\ r\leqq k\leqq m_j\\ 1\leqq l\leqq r}} b_j^k q_l\Big)_{1\leqq i,j\leqq r}$$

if we denote by $q_j=q_j(x',\eta')$ and $b_j^k=b_j^k(x',\eta')$ the principal symbol of order j and m_j-k of $Q_j(0,x',D')$ and $B_j^k(x',D')$ respectively. (Cf. Volevič [8] and Gårding-Kotaké-Leray [2].)

Then we can apply the results of Sato-Kawai-Kashiwara [6] to (1) and find the compatibility conditions which $\mu_j(x')$ should satisfy if the above boundary value problem is not solvable as far as $\Delta(x',\eta')$ satisfies the conditions of Theorem 1 in Sato-Kawai-Kashiwara [6]. Note that we can treat the case where $\Delta(x',\eta')$ has constant multiplicity (>1). The authors believe that this is a crucial point because the ellipticity of $\mathcal{M}$ does not give any informations about the multiplicity of $\Delta(x',\eta')$, e.g. $P(x,D)=(D_1^2+\cdots+D_n^2)^2$.

Thus we have encountered again the advantages of the employment of pseudo-differential operators of infinite order even in such an elementary problem as the boundary value problem for single elliptic equation.

Sketch of the proof of Theorem 1 in [4]. The proof is devided into two steps. Note that we need only the assumption of ellipticity in the first step (Proposition 2) and only the non-characteristic assumption in the second step (Proposition 3). This observation allows us to bypass many technical difficulties in the course of the proof.

We begin our discussions by introducing the sheaf $\mathcal{C}_{N/X}$ on S_N^*X. Using the comonoidal transformation $\widetilde{{}^NX^*}$ of X with center N, we define

$C_{N/X}$ by $\mathcal{H}^n_{S^*_N X}(\pi^{-1}_{N/X}\mathcal{O}_X)^a \otimes \omega_N$. Here ω_N is the orientation sheaf of N. The sheaf $C_{N/X}$ is seen to be a left $\mathcal{P}_X|_{S^*_N X}$-Module.

Now, we have the following $\mathcal{D}_M$-linear homomorphisms:

$$
(4)\qquad
\begin{array}{ccc}
\boldsymbol{R}\Gamma_{S^*_N M}(\pi^{-1}_{N/M}\mathcal{A}_M)^a \otimes \omega_{N/M} & \leftarrow & \boldsymbol{R}q_!\,(C_{N/X}|_{S^*_N X - \sqrt{-1}S^*M}) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\Gamma_{S^*_N M}(\pi^{-1}_{N/M}\mathcal{B}_M)^a \otimes \omega_{N/M} & \rightarrow & \boldsymbol{R}q_*(C_{N/X}|_{S^*_N X - \sqrt{-1}S^*M}).
\end{array}
$$

Hence we have

$$
(5)\qquad
\begin{array}{l}
\boldsymbol{R}\Gamma_{S^*_N M}(\pi^{-1}_{N/M}\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, \mathcal{A}_N))^a \otimes \omega_{N/M} \\
\qquad\downarrow \\
\boldsymbol{R}\Gamma_{S^*_N M}(\pi^{-1}_{N/M}\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, \mathcal{B}_N))^a \otimes \omega_{N/M} \\
\qquad\qquad \leftarrow \boldsymbol{R}q_!\,(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, C_{N/X})|_{S^*_N X - \sqrt{-1}S^*M}) \\
\qquad\qquad\qquad\downarrow \\
\qquad\qquad \rightarrow \boldsymbol{R}q_*(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, C_{N/X})|_{S^*_N X - \sqrt{-1}S^*M})
\end{array}
$$

Since $\mathcal{M}$ is elliptic, the left column in (5) gives clearly an isomorphism. Moreover the ellipticity of $\mathcal{M}$ implies that $\mathrm{Supp}\,(\mathcal{P}_X \otimes_{\mathcal{D}_X} \mathcal{M}) \rightarrow S^*_N M$ is a proper map. Therefore the right column in (5) gives an isomorphism, since $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, C_{N/X}) = \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}_X}(\mathcal{P}_X \otimes_{\mathcal{D}_X} \mathcal{M}, C_{N/X})$.

On the other hand we have the following

Proposition 2. *If $\mathcal{M}$ is an elliptic system of linear differential equations on M, then the horizontal arrows in* (5) *are isomorphisms. Here we need not assume that N is non-characteristic with respect to $\mathcal{M}$.*

This proposition is proved by the reduction to the case where $\mathcal{M}$ is a single equation. Such a reduction is possible since we do not need the non-characteristic assumption in this proposition.

Now, we go on to the second step of the proof, i.e. we show the following

Proposition 3. *Let $\mathcal{M}$ be a system of pseudo-differential equations with respect to which N is non-characteristic. Then we have the following isomorphism:*

$$
(6)\qquad \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}_X}(\mathcal{M}, C_{N/X}) \cong \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}_X}(\mathcal{M}, \mathcal{P}_{X \leftarrow Y}) \overset{L}{\underset{p^{-1}\mathcal{P}_Y}{\otimes}} p^{-1}C_N.
$$

It is clear that Proposition 2 and Proposition 3 immediately prove Theorem 1. On the other hand, we can show that the sheaf $C_{N/X}$ is isomorphic to the sheaf of microfunctions with holomorphic parameters outside $S^*_Y X \cap S^*_N X$. Hence, Proposition 3 is equivalent to the corresponding proposition about the sheaf of microfunctions with holomorphic parameters. Before stating the proposition we make clear the meaning of the sheaf of microfunctions with holomorphic parameters.

Let $f: X \rightarrow Y$ be a smooth holomorphic map with fiber dimension d. Suppose that Y is a complexification of a real analytic manifold N and set $M = f^{-1}(N)$. Denoting by $\pi_{M/X}: {}^M\widetilde{X^*} = {}^N\widetilde{Y^*} \underset{Y}{\times} X \rightarrow X$ the comonoidal transformation of X with center M, $\tilde{C}_M$ is, by definition, $\mathcal{H}^d_{S^*_M X}(\pi^{-1}_{M/X}\mathcal{O}_X)^a \otimes \omega_N$. $\tilde{C}_M$ is, in fact, the subsheaf of C_M consisting of the microfunctions on M satisfying the Cauchy-Riemann equation along the fibers of f. $\tilde{C}_M$ is a $(\mathcal{P}_X|_{S^*_M X})$-Module defined on $S^*_M X = M \underset{N}{\times} \sqrt{-1}S^*N$. We denote by the same letter f the projection $S^*_M X \to \sqrt{-1}S^*N$.

Now using the above notations we state the following proposition which is equivalent to Proposition 3.

Proposition 4. *Let $\mathcal{M}$ be a system of pseudo-differential equations on X with respect to which f is non-characteristic, that is,* Supp $\mathcal{M} \cap M \underset{Y}{\times} P^*Y \to P^*Y$ *is a finite morphism. Then we have the following isomorphism.*

$$(7)\qquad \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}_X}(\mathcal{M}, \tilde{C}_M) \cong \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}_X}(\mathcal{M}, \mathcal{P}_{X\leftarrow Y}) \overset{L}{\underset{f^{-1}\mathcal{P}_Y}{\otimes}} f^{-1}C_N.$$

This proposition is proved by the reduction to the case where $d=1$ and $\mathcal{M}$ is a single equation.

Errata in our previous note [4].

The isomorphism (1) should be replaced by the following:

$$\boldsymbol{R}\Gamma_{S^*_N M}(\pi^{-1}_{N/M}\mathcal{S})^a \otimes \omega_{N/M}$$
$$\cong \boldsymbol{R}q_* \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{P}_{X\leftarrow Y}|_{S^*_N X}) \overset{L}{\underset{p^{-1}\mathcal{P}_Y}{\otimes}} p^{-1}C_N)$$

Note that the right hand side reduces to

$$\boldsymbol{R}t_* \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}_N}(p_*(\mathcal{P}_{Y\to X} \underset{\mathcal{D}_X}{\otimes} \mathcal{M}|_{S^*_N X}), C_N)\,[-d]$$

when the restriction of q to $Z=\text{Supp}\,\mathcal{M} \cap S^*_N X$ decomposes into the composite of the mapping p and the mapping t from $p(Z)$ to $S^*_N M$.

The left hand sides of (2), (3) and (5) should be corrected according to the above correction. The results in all the examples of [4] remain to be valid.

References

[1] Egorov, Yu. V., and V. A. Kondratev: The oblique derivative problem. Math. Sbornik, **78**, 148–176 (1969) (in Russian).

[2] Gårding, L., T. Kotaké, and J. Leray: Problème de Cauchy. Ibis et VI. Bull. Soc. Math. Fr., **92**, 263–361 (1964).

[3] Hörmander, L.: Pseudo-differential operators and non-elliptic boundary value problems. Ann. Math., **83**, 129–209 (1966).

[4] Kashiwara, M., and T. Kawai: On the boundary value problem for elliptic system of linear differential equations. I. Proc. Japan Acad., **48**, 712–715 (1972).

[5] Sato, M., T. Kawai, and M. Kashiwara (*S-K-K*): Microfunctions and pseudo-differential equations (to appear in Proceedings of Katata Conference).

[6] ——: On the structure of single linear pseudo-differential equations. Proc. Japan Acad., **48**, 643–646 (1972).

[7] Sjöstrand, J.: Operators of principal type with interior boundary conditions (to appear in Acta Math.).

[8] Volevič, L. R.: On general systems of differential equations. Dokl. Akad. Nauk U. S. S. R., **132**, 20–23 (1960) (in Russian).