35. *Two Theorems on Mix-Relativization*

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In this paper we shall consider "relativization by a set of unary predicate symbols" and state two theorems about it, which can be considered as extensions of the usual relativization theorem (cf. Motohashi [2]) and one sorted reductions of Feferman's many sorted interpolation theorems (Theorem 4.2 and Theorem 4.4 in Feferman [1]). The key ideas of our proofs of these theorems have already been appeared in [2] although their proofs themselves will be omitted in this paper, and details will be published elsewhere.

Let L be a first order finitary or infinitary logic ($L_{\omega\omega}$ or $L_{\omega_1\omega}$ in [1]), $\boldsymbol{U}=\{U_i\}_{i\in I}$ a set of unary predicate symbols which do not appear in L and $\boldsymbol{L}$ the first order logic obtained from L by adding every predicate symbol in $\boldsymbol{U}$. For the sake of covenience, we assume that L has neither individual constant symbols nor function symbols. Let A be a formula in $\boldsymbol{L}$ and B in L. Then we say that "A is a *mix-relativization formula* of B (by $\boldsymbol{U}$)" or "A is obtained from B through *mix-relativization* (by $\boldsymbol{U}$)" if A is obtained from B by relativization some occurrences of quantifiers of B by predicate symbols in $\boldsymbol{U}$. If every occurrence of quantifiers in B is relativized by a predicate symbol in $\boldsymbol{U}$, we say that A is a *total mix-relativization formula* of B. For example, the formula $(\forall u)(U_i(u)\supset(\exists v)(U_j(v)\wedge C(u,v))$ is a mix-relativization formula of $(\forall u)(\exists v)C(u,v)$, where $i,j\in I$ and $C(x,y)$ is a formula in L. Moreover if $C(x,y)$ has no occurrence of quantifiers, then that formula is a total mix-relativization formula of $(\forall u)(\exists v)C(u,v)$. If A is a (total) mix-relativization formula of a formula in L, we simply say that A is a (total) mix-relativization formula. For each mix-relativization formula A, let $I(A)$, $U_n(A)$ and $E_x(A)$ be the set of all $i\in I$ such that U_i appears in A, the set of all $i\in I$ such that U_i appear negatively in A and the set of all $i\in I$ such that U_i appear positively in A respectively (cf. [1]). Hence $I(A)=U_n(A)\cup E_x(A)$. For example, if A is the formula above mentioned, then $U_n(A)=\{i\}$ and $E_x(A)=\{j\}$. Notice that if A is a formula in L, then A is a mix-relativization formula and $I(A)=U_n(A)=E_x(A)=\phi$. Also if A is a total mix-relativization formula of B and $I(A)=\{i\}$, then $A=B^{U_i}$, i.e. A is the relativization formula of B by U_i in the usual sense. Then we have the following two theorems.

Theorem I. *Suppose I_0 and I_1 are subsets of I, A and B are total* *mix-relativization formulas and* $\{x_i\}_{i\in I_1}$ *is a set of free variables such that every free variable which occurs either in A or in B belongs to it.*

If $\{(\exists u)U_i(u)\}_{i\in I_0}, \{U_i(x_i)\}_{i\in I_1} \vdash_L A\supset B$, *then there is a total mix-relativization formula C in* **L** *satisfying the following four conditions* 1)–4):

1) $$\{(\exists u)U_i(u)\}_{i\in I_0},\quad \{U_i(x_i)\}_{i\in I_1} \vdash_L \quad A\supset C$$

and

$$\{(\exists u)U_i(u)\}_{i\in I_0},\quad \{U_i(x_i)\}_{i\in I_1} \vdash_L \quad C\supset E.$$

2) *Every predicate symbol of C in* **L** *except* $U_i, i\in I_1$, *occurs both in A and in B.*

3) *Every free variable of C belongs to* $\{x_i\}_{i\in I_1}$.

4) $$U_n(C)\subsetneqq U_n(A) \text{ and } E_x(C)\subsetneqq E_x(B).$$

Theorem II. *Suppose* I_0 *and* I_1 *are subsets of I, A is a total mix-relativization formula, B is a mix-relativization formula and,* $\{y_i\}_{i\in I_0}$ *and* $\{x_i\}_{i\in I_1}$ *are two sets of free variables such that every free variable in A belongs to* $\{x_i\}_{i\in I_1}$.

If $\{(\exists u)U_i(u)\}_{i\in I_0}, \{U_i(x_i)\}_{i\in I_1} \vdash_L A\supset B$, *then there is a total mix-relativization formula C in* **L** *satisfying the following four conditions* 5)–8):

5) $$\{U_i(y_i)\}_{i\in I_0},\quad \{U_i(x_i)\}_{i\in I_1} \vdash_L \quad A\supset C$$

and

$$\{U_i(y_i)\}_{i\in I_0},\quad \{U_i(x_i)\}_{i\in I_1} \vdash_L \quad C\supset B.$$

6) *Every predicate symbol of C in* **L** *occurs both in A and in B.*

7) *Every free variable of C belongs to* $\{y_i\}_{i\in I_0}\cup\{x_i\}_{i\in I_1}$.

8) $$U_n(C)\subsetneqq U_n(B) \quad \text{and} \quad E_x(C)\subsetneqq E_x(A).$$

If $\Gamma,\ U_i(x)\vdash_L A(x)\supset B$ and x appears neither in Γ nor in B, then $\Gamma\vdash_L (\exists u)(U_i(u)\wedge A(u))\supset B$. Notice that $U_n((\exists u)(U_i(u)\wedge A(u))) = U_n(A(x))$ but $E_x((\exists u)(U_i(u)\wedge A(u)) = E_x(A(x))\cup\{i\}$. If $\Gamma,\ U_i(x)\vdash_L A\supset B(x)$ and x appears neither in Γ nor in A, then $\Gamma\vdash_L A\supset(\forall u)(U_i(u)\supset B(u))$. Notice that $E_x((\forall u)(U_i(u)\supset B(u)))=E_x(B(x))$ but $U_n((\forall u)(U_i(u)\supset B(u)))=U_n(B(x))\cup\{i\}$. These two facts show us that in Theorem I we can add the condition that every free variable of C occurs both in A and in B but can not in Theorem II.

Remark 1. *Let D and E be sentences in L and* $U\in \boldsymbol{U}$. *Suppose* $(\exists u)U(u)\vdash_L D^U\supset E$. *Then by Theorem II, we have a sentence C in L such that* $(\exists u)U(u)\vdash_L D^U\supset C^U$, $(\exists u)U(u)\vdash_L C^U\supset E$ *and* $U_n(C^U)\subsetneqq U_n(E)=\phi$. *This means that C is an existential sentence and*"$\vdash_L D\supset C$" *and* "$\vdash_L C\supset E$" *hold. This is the usual relativization theorem* (cf. [2]).

Remark 2. *We use Feferman's terminology in* [1]. *Let* L_m *be a many sorted logic and, A and B are two sentences in* L_m *such that* "$\vdash_{L_m} A\supset B$" *holds. Let* A^* *and* B^* *be their one sorted reductions in* **L**, *hence we can consider* A^* *and* B^* *as two mix-relativization sentences in*

L. *Let* $I_0 = I(A^*) \cup I(B^*)$. *Then we have*

$$\{(\exists u)U_i(u)\}_{i \in I_0} \vdash_L A^* \supset B^*.$$

By Theorem I, there is a total mix-relativization sentence C_1 *satisfying* 1)–4) *in Theorem I. Since* C_1 *is a total mix-relativization,* $C_1 = C^*$ *for some sentence* C *in* L_m. *This* C *satisfies*: (i) *every predicate in* C *occurs both in* A *and in* B, (ii) $\vdash_{L_m} A \supset C$ *and* $\vdash_{L_m} C \supset B$, (iii) $U_n(C) \subseteq U_n(A)$ *and* $E_x(C) \subseteq E_x(B)$. *This is the Feferman's many sorted interpolation theorem, i.e. Theorem* 4.2 *in* [1].

Remark 3. *Suppose* A *and* B *are sentences in a many sorted logic* L_m *and* $I_0 \subseteq I$. *If* $\vdash_{L_m} A \supset B$, $E_x(A) \subseteq I_0$ *and* $U_n(B) \subseteq I_0$, *then by Theorem II, we have a formula* C *in* L_m *satisfying*: (i) $\vdash_{L_m} A \supset C$ *and* $\vdash_{L_m} C \supset B$, (ii) $U_n(C) \subseteq I_0$ *and* $E_x(C) \subseteq I_0$. *This is Theorem* 4.4 *in* [1].

References

[1] S. Feferman: Lectures on Proof Theory. Proceedings of the Summer School in Logic, Leeds (1967). Lecture Notes in Math. No. 70, Springer-Verlag, 1–107 (1968).

[2] N. Motohashi: An extended relativization theorem (to appear in J. Math. Soc. Japan, **25** (1973)).