34. *Continuity of the Map $S\to|S|$ for Linear Operators*[*)]

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This note is concerned with the continuity of the map $|\cdot|$ from $B(H,H')$ to $B_{sa}(H)$ given by $|S|=(S^*S)^{1/2}$; here $B(H,H')$ denotes the set of all bounded linear operators on a Hilbert space H to another Hilbert space H', and $B_{sa}(H)$ the set of all bounded selfadjoint operators in H. We shall prove the following results.

I. *The map $|\cdot|$ is almost Lipschitz-continuous,in the sense that*

$$\||S|-|T|\|\leq\frac{2}{\pi}\|S-T\|\left(2+\log\frac{\|S\|+\|T\|}{\|S-T\|}\right),$$

where $\|\cdot\|$ denotes the operator norm.

II. *If both H and H' are infinite-dimensional, the map $|\cdot|$ is not Lipschitz-continuous in the operator norm, even when $H'=H$ and $|\cdot|$ is restricted on $B_{sa}(H)$.*

III. *For each integer $n\geq 1$, there is a holomorphic family of operators $S(t)\in B_{sa}(H)$, $-1<t<1$, where H is a finite-dimensional Hilbert space, with the following properties.* (i) $0<|S(t)|<2I$, (ii) $\|dS(t)/dt\|<1$, *and* (iii) $\|[d|S(t)|/dt]_{t=0}\|>n^2$. *Note that $|S(\cdot)|$ is also holomorphic.*

IV. *There exists a family $T(t)$, $-1<t<1$, of selfadjoint operators in a separable Hilbert space H such that $T(t)^{-1}$ exists as a bounded operator, $T(t)^{-1}$ is norm-continuously differentiable in $t\in(-1,1)$, but $|T(t)^{-1}|$ is not weakly differentiable at $t=0$.*

Remarks. 1. Propositions I and II answer some questions that appear to have been open, see e.g. Reed and Simon [1, p. 197].

2. In II it suffices to consider the special case mentioned at the end. The result for this special case is, however, a direct consequence of III.

3. IV answers a question raised by Cooper [2].

4. It seems difficult to construct a twice differentiable family $T(t)^{-1}$ with properties similar to those stated in IV. The reason is that $\|A\|$ used in (8) below grows very fast with n. Thus it is not known to the author whether or not the continuous differentiability of $T(t)^{-1}$ can be replaced by a higher order differentiability or even by analyticity.

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5. If at least one of H and H' is finite-dimensional, the map $|\cdot|$ is Lipschitz continuous in the operator norm. This follows from a more general theorem, due to W. Kahan, that the map is Lipschitz-continuous in the Hilbert-Schmidt norm (even in the infinite-dimensional case).

Proof of I. We use the well-known formula (see e.g. [3, p. 285])

$$\pi|S|-\pi|T|=\int_0^\infty \lambda^{1/2}[(|T|^2+\lambda)^{-1}-(|S|^2+\lambda)^{-1}]d\lambda. \tag{1}$$

We split the integral in (1) into three parts, $\int_0^\alpha$, $\int_\alpha^\beta$, and $\int_\beta^\infty$, where $\alpha=\|S-T\|^2$ and $\beta=(\|S\|+\|T\|)^2$. Since

$$\|(|T|^2+\lambda)^{-1}-(|S|^2+\lambda)^{-1}\|\le\max\{\|(|T|^2+\lambda)^{-1}\|,\|(|S|^2+\lambda)^{-1}\|\}\le\lambda^{-1},$$

we have

$$\left\|\int_0^\alpha\right\|\le\int_0^\alpha\lambda^{-1/2}d\lambda=2\alpha^{1/2}=2\|S-T\|. \tag{2}$$

In the remaining integrals, we rewrite the integrand in the form

$$\lambda^{1/2}(|T|^2+\lambda)^{-1}(|S|^2-|T|^2)(|S|^2+\lambda)^{-1}. \tag{3}$$

Since (3) is majorized in norm by $\lambda^{-3/2}\||S|^2-|T|^2\|$, we have

$$\left\|\int_\beta^\infty\right\|\le2\||S|^2-|T|^2\|\beta^{-1/2}\le2\|S-T\|; \tag{4}$$

note that

$$\begin{aligned}\||S|^2-|T|^2\|=\|S^*S-T^*T\|&\le\|S^*\|\|S-T\|+\|S^*-T^*\|\|T\|\\&\le(\|S\|+\|T\|)\|S-T\|=\beta^{1/2}\|S-T\|.\end{aligned}$$

In the integral $\int_\alpha^\beta$, we further replace $|S|^2-|T|^2$ in (3) by $S^*S-T^*T=T^*(S-T)+(S^*-T^*)S$ and use the estimates

$$\begin{aligned}\|(|T|^2+\lambda)^{-1}T^*\|&=\|T(|T|^2+\lambda)^{-1}\|\\&=\||T|(|T|^2+\lambda)^{-1}\|\le1/2\lambda^{1/2},\end{aligned}\qquad\text{etc.}$$

Then we obtain

$$\begin{aligned}\left\|\int_\alpha^\beta\right\|&\le\|S-T\|\int_\alpha^\beta\lambda^{-1}d\lambda=\|S-T\|\log(\beta/\alpha)\\&=2\|S-T\|\log\frac{\|S\|+\|T\|}{\|S-T\|}.\end{aligned} \tag{5}$$

Collecting (2), (4), and (5), we obtain the desired result of I.

Proof of III. We use an example due to McIntosh [4], which appears to be an inexhaustible source of counter-examples of this type (see also [5]). In [4] it is shown that there exist selfadjoint operators A, B in a finite dimensional Hilbert space H, with A invertible, such that

$$\|[A,B]\|<1,\qquad\|[|A|,B]\|>n^2+2, \tag{6}$$

where [,] denotes the commutator. We shall normalize A,B so that

$$\|A^{-1}\|=1,\qquad\text{hence }|A|\ge I,\ \|A\|\ge1. \tag{7}$$

Set

$$S(t)=A^{-1}+ib\sin(t/b)[A,B],\qquad-1<t<1, \tag{8}$$

where b is a constant such that

$$0<b<1/\|A\|. \tag{9}$$

$S(t)$ is selfadjoint and holomorphic in t. If $0 \neq u \in H$, we have

$$| \| |S(t)| u \| - \| A^{-1} u \| | = | \| S(t) u \| - \| A^{-1} u \| | \leq \| [S(t) - A^{-1}] u \| \leq b \| [A, B] u \| < \| A \|^{-1} \| u \|$$

by (8), (6), and (9). Hence

$$0 \leq \| A^{-1} u \| - \| A \|^{-1} \| u \| < \| |S(t)| u \| < \| A^{-1} u \| + \| A \|^{-1} \| u \| \leq 2 \| u \|$$

by (7). This proves (i). (ii) follows from $dS(t)/dt = i \cos(t/b)[A, B]$ and (6).

It remains to prove (iii). We note that $|S(t)|$ is also holomorphic in t because $S(t)$ has no eigenvalue 0 (see [3, p. 416]. Thus

(10) $$|S(t)| = |A|^{-1} + tC + O(t^2), \qquad \text{as } |t| \to 0.$$

Since $S(t) = A^{-1} + it[A, B] + O(t^2)$, comparison of the coefficients of t in the expansion for $|S(t)|^2 = S(t)^2$ gives

(11) $$|A|^{-1} C + C |A|^{-1} = iA^{-1}[A, B] + i[A, B]A^{-1} = i(ABA^{-1} - A^{-1}BA).$$

Since $|A|^{-1} > 0$, (11) determines C uniquely (see Heinz [6]). C is given by

(12) $$C = i(ABU - UBA) = i[A, B]U + iU[A, B] - i[|A|, B],$$

where $U = \operatorname{sign} A$ is a unitary operator. Indeed it is easy to verify that (12) is a solution of (11), using $|A|^{-1} U = A^{-1}$, $|A|^{-1} A = U$, etc. Now (6) and (12) show that $\| C \| > n^2$. This proves (iii).

Proof of IV. The desired counter-example can be constructed as the direct sum $T(t) = \bigoplus_{n=1}^{\infty} n T_n(t)$ in the space $H = \bigoplus_{n=1}^{\infty} H_n$, where H_n and $T_n(t) = S_n(t)^{-1}$ are the H and $S(t)^{-1}$ of III. Since $S_n(t)$ is invertible by III, (i), $T(t)$ is well defined as a selfadjoint operator in H, with

(13) $$T(t)^{-1} = \bigoplus_{n=1}^{\infty} n^{-1} S_n(t).$$

$T(t)^{-1}$ is a bounded selfadjoint operator for each $t \in (-1, 1)$, since $\| S_n(t) \| \leq 2$ by III, (i).

To show that $T(t)^{-1}$ is norm-continuously differentiable, set $R(t) = \bigoplus n^{-1} dS_n(t)/dt$. Since $\| dS_n(t)/dt \| \leq 1$ by III, (ii), $R(t)$ is also bounded and norm-continuous in t. Then it is easy to show that $T(t)^{-1}$ is an indefinite integral of $R(t)$, so that $T(t)^{-1}$ is norm-continuously differentiable.

On the other hand, III, (iii) shows that there is $u_n \in H_n \subset H$ such that $| [d(|T(t)^{-1}| u_n, u_n)/dt]_{t=0} | = n^{-1} | [d(|S_n(t)| u_n, u_n)/dt]_{t=0} | > n \| u_n \|^2$. This shows that $|T(t)^{-1}|$ is not weakly differentiable at $t = 0$.

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