73. *On a Semantics for Non-Classical Logics*

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In [2] and [3], Ono showed some incompleteness results on two types of semantics for the intermediate predicate logics, that is, the algebraic and the Kripke-type. More precisely, he proved that there exist many intermediate predicate logics without characteristic sets of algebraic models, and that there exist those without characteristic Kripke models. This situation is more serious in the case of the modal predicate logics. In fact he proved the existence of a modal predicate logic having neither characteristic sets of algebraic models nor characteristic Kripke models.

Thus the existing semantical methods proved incomplete in the above sense. Therefore, some new type of semantics is required since the semantical method is indispensable for the study of logics.

This note proposes one of such semantics that contains the algebraic semantics as well as Kripke-type one as special cases. Our new semantics is obtained by combining these two types of semantics quite naturally.

Some applications of this semantics for the intermediate logics will be studied in a paper to appear in near future.

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§ 1. Semantics for the intermediate logics.

In this section we describe our semantics for the intermediate logics. Our basic language $\mathcal{L}$ is a usual one (see e.g. Ono [4]).

Definition 1.1. 1) *A pseudo-Boolean algebra P is called $\lambda^<$-complete, if there always exist $\bigcup_{t\in T} a_t$ and $\bigcap_{t\in T} a_t$ for any subset $\{a_t\}_{t\in T}$ of P such that $\overline{\overline{T}}<\lambda$.*

2) *A subalgebra P' of a pseudo-Boolean algebra P is said to be $\lambda^<$-complete, if for any subset $\{a_t\}_{t\in T}$ of P' such that $\overline{\overline{T}}<\lambda$ $\bigcup_{t\in T} a_t$, $\bigcap_{t\in T} a_t$ belong to P' whenever they exist in P.*

Definition 1.2. *By a model we mean a triple $(M, V; P)$ satisfying the following conditions;*

1) *M is a non-empty partially ordered set with the order relation $\leqslant_M$,*

2) *V is a mapping from M to the power set of some set such that $V(a)$ is non-empty for any $a \in M$ and $V(a)\subseteq V(b)$ if $a \leqslant_M b$,*

3) *P is a non-degenerate $\kappa(M, V)^<$-complete pseudo-Boolean*

algebra, where $\kappa(M, V)$ denotes the smallest cardinal which is greater than $\overline{\overline{V(a)}}$ and $\overline{\overline{\{b \in M : a \leqslant_M b\}}}$ for every $a \in M$.

Now let $(M, V; P)$ be a model. We define an $(M, V; P)$-valuation. Let $\mathcal{L}[V]$ be the language obtained from $\mathcal{L}$ by adding the individual constants $\bar{\xi}$ (the name of ξ) for each element ξ of $\bigcup_{a \in M} V(a)$. An $(M, V; P)$-valuation W is a mapping from the cartesian product of the set of closed formulae of $\mathcal{L}[V]$ and M into P, which satisfies the following conditions;

1) for n-adic $(n \geqslant 0)$ predicate variable $p^{(n)}$, $W(p^{(n)}\bar{\xi}_1 \cdots \bar{\xi}_n, a) \leqslant W(p^{(n)}\bar{\xi}_1 \cdots \bar{\xi}_n, b)$ if $a \leqslant_M b$ and $\langle \xi_1, \cdots, \xi_n \rangle \in V(a)^n$,

2) $W(A \wedge B, a) = W(A, a) \cap W(B, a)$,

3) $W(A \vee B, a) = W(A, a) \cup W(B, a)$,

4) $W(\neg A, a) = \bigcap_{a \leqslant_M b} - W(A, b)$,

5) $W(A \rightarrow B, a) = \bigcap_{a \leqslant_M b} (W(A, b) \supset W(B, b))$,

6) $W(\exists x A, a) = \bigcup_{\xi \in V(a)} W(A_x[\bar{\xi}], a)$,

7) $W(\forall x A, a) = \bigcap_{a \leqslant_M b} \bigcap_{\xi \in V(b)} W(A_x[\bar{\xi}], b)$.

A closed formula A of $\mathcal{L}$ is said to be valid in a model $(M, V; P)$ if, for any $(M, V; P)$-valuation W and for any $a \in M$, $W(A, a) = 1_P$, where 1_P denotes the greatest element of P. An arbitrary formula is said to be valid in $(M, V; P)$ if its closure is valid in it. We write $L(M, V; P)$ for the set of all formulae of $\mathcal{L}$ valid in the model $(M, V; P)$.

Theorem 1.2. *If $(M, V; P)$ is a model, then $L(M, V; P)$ is closed under modus ponens, generalization and substitution and contains the intuitionistic predicate logic LJ.*

Theorem 1.3. 1) *If M is a singleton $\{a\}$, then $L(\{a\}, V; P)$ is nothing but $L^+(P, V(a))$ in* [4] *and $(\{a\}, V; P)$ can be regarded as an algebraic model in the sense of* [4].

2) *If P is the two-valued Boolean algebra S_1, then $(M, V; S_1)$ is a usual Kripke model and therefore $L(M, V; S_1)$ is no other than $L(M, V)$ in* [4].

Thus, as stated before, the semantics introduced above is a generalization of the two types of semantics.

Lemma 1.4. *Let $(M, V; P)$ be a model. If P' is a $\kappa(M, V)^{<}$-complete subalgebra of P, then $L(M, V; P) \subseteq L(M, V; P')$.*

Corollary 1.5. *If there exists $a \in M$ such that $\overline{\overline{V(a)}} \geqslant \aleph_0$, then $L(M, V; P) \subseteq LK$ and hence $L(M, V; P)$ is an intermediate predicate logic in the sense of* [4].

Proof. First, $L(M, V; P) \subseteq L(M, V; S_1)$ by Lemma 1.4. By the assumption and Theorem 3.6 in [4], we have that $L(M, V; S_1) \subseteq LK$. Hence $L(M, V; P) \subseteq LK$.

§ 2. Semantics for the modal logics.

Here we describe a modal version of our semantics. Our modification follows the method introduced by E. J. Lemmon in [1]. Here $\mathcal{L}$ consists of the individual and the predicate variables and the logical constants $\vee, \neg, \Diamond, \exists$.

Definition 2.1. *By a model we mean a quadruple* $(\mathfrak{M}, V; Q; T)$ *such that*

1) $\mathfrak{M}=(M, R)$ *is a relational structure, i.e.* M *is a non-empty set and* R *is a binary relation on* M,

2) V *is a mapping from* M *to the power set of some set such that* $V(a)$ *is non-empty for any* $a \in M$ *and* $V(a) \subseteq V(b)$ *if* Rab,

3) Q *is a mapping from* M *into* T,

4) T *is a* $\kappa(\mathfrak{M}, V)^{<}$*-complete modal algebra, i.e.* T *is a* $\kappa(\mathfrak{M}, V)^{<}$*-complete Boolean algebra with additional operator* $\boldsymbol{P}$ *satisfying the axiom* $\boldsymbol{P}(s \cup t)=\boldsymbol{P}s \cup \boldsymbol{P}t$, *where* $\kappa(\mathfrak{M}, V)$ *is the smallest cardinal greater than* $\overline{\overline{V(a)}}$ *and* $\overline{\overline{\{b \in M : Rab\}}}$ *for any* $a \in M$.

An $(\mathfrak{M}, V; Q; T)$-valuation W is defined as a mapping satisfying the following conditions;

1) for n-adic $(n \geqslant 0)$ predicate variable $p^{(n)}$, $W(p^{(n)}\xi_1 \cdots \xi_n, a) \in T$ if $\langle \xi_1, \cdots, \xi_n \rangle \in V(a)^n$ and $a \in M$,

2) $W(A \vee B, a)=W(A, a) \cup W(B, b)$,

3) $W(\neg A, a)=-W(A, a)$,

4) $W(\Diamond A, a)=\boldsymbol{P}(\bigcup_{Rab} W(A, b)) \cup Q(a)$,

5) $W(\exists x A, a)=\bigcup_{\xi \in V(a)} W(A_x[\xi], a)$, where $p^{(n)}\xi_1 \cdots \xi_n, A \vee B, \neg A$ etc. are closed formulae of $\mathcal{L}[V]$, the language obtained from $\mathcal{L}$ as before.

The validity in a model $(\mathfrak{M}, V; Q; T)$ is defined in the same way as in the case of the intermediate logics. $L(\mathfrak{M}, V; Q; T)$ denotes the set of all formulae of $\mathcal{L}$ valid in a model $(\mathfrak{M}, V; Q; T)$.

A model $(\mathfrak{M}, V; Q; T)$ introduced above is, so to say, a C_2^*-model, where C_2^* is the modal predicate logic corresponding to C_2 in [1]. We have that $C_2^* = \cap L(\mathfrak{M}, V; Q; T)$, where the conjunction is taken over all the models.

If we impose some conditions on a model $(\mathfrak{M}, V; Q; T)$, we get a D_2^*-model, an E_2^*-model, a $T(C)^*$-model, an S_4^*-model etc., where D_2^*, E_2^*, $T(C)^*$ and S_4^* are the modal predicate logics corresponding to D_2, E_2, $T(C)$ and S_4 in [1] respectively.

Theorem 2.2. *Let* $(\mathfrak{M}, V; Q; T)$ *be a model and* $L=L(\mathfrak{M}, V; Q; T)$,

1) $L \supseteq D_2^*$ *iff, for any* $a \in M$, *either there exists* $b \in M$ *such that* Rab *and* $\boldsymbol{P}1 \cup Q(a)=1$, *or* $\boldsymbol{P}0 \cup Q(a)=1$,

2) $L \supseteq E_2^*$ *iff, for any* $a \in M$, *either* Raa *and* $t \leqslant \boldsymbol{P}t \cup Q(a)$ *for every* $t \in T$, *or* $\boldsymbol{P}0 \cup Q(a)=1$,

3) $L \supseteq T(C)^*$ *iff* $Q(a)=0$ *for every* $a \in M$ *and* $\boldsymbol{P}0=0$,

4) $L \supseteq S_4^*$ *iff* $Q(a)=0$ *for every* $a \in M$, R *is reflexive and transitive*

and T is a closure algebra i.e. $\boldsymbol{P}$ *satisfies additional axioms*; $\boldsymbol{P}0=0$, $\boldsymbol{PP}t=\boldsymbol{P}t$ *and* $t\leqslant \boldsymbol{P}t$ *for every* $t\in T$.

Moreover, as in the case of C_2^*, the above four logics can be proved complete for their models.

As a consequence of this theorem, we obtain the analogy of Theorem 1.3 and therefore our semantics for the modal logics is also a generalization of the algebraic semantics and Kripke-type one for the modal logics.

References

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