72. *On Banach-Steinhaus Theorem*

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The theory of ranked space is a new and constructive method of the mathematical analysis, which has been investigated by K. Kunugi since 1954 [1]. We proved the closed graph theorem in ranked spaces with some conditions [4]. And now, in this note we shall prove the Banach-Steinhaus theorem in ranked spaces, whose neighbourhoods need not be open. Throughout this note, $g, f, \cdots$ will denote points of a ranked space, $U_i, V_i, \cdots$ neighbourhoods of the origin with rank i, $\{U_{r_i}\}, \{V_{r_i}\}, \cdots$ fundamental sequences of neighbourhoods with respect to the origin and $U_i(g), V_i(g), \cdots$ neighbourhoods of the point g with rank i.

Let a linear space E be a complete ranked space with indicator ω_0, which satisfies the following conditions.

(E, 1) (1) For any neighbourhood U_i, the origin belongs to U_i.

(2) For any U_i and V_j, there is a W_k such that $W_k \subseteqq U_i \cap V_j$.

(3) For any neighbourhood U_i and for any integer n, there is an m such that $m \geqq n$ and $U_m \subseteqq U_i$.

(4) The E is the neighbourhood of the origin with rank zero.

(E, 2) The following conditions are the modification of the Washihara's conditions [3].

(R, L_1) For any $\{U_{r_i}\}$ and $\{V_{r'_i}\}$, there is a $\{W_{r''_i}\}$ such that $U_{r_i} + V_{r'_i} \subseteqq W_{r''_i}$.

(R, L_2)′ (1) For any $\{U_{r_i}\}$ and $\lambda > 0$, there is a $\{V_{r'_i}\}$ such that $\lambda U_{r_i} \subseteqq V_{r'_i}$.

(2) For any $\{U_{r_i}\}$ and $\{\lambda_i\}$ with $\lim \lambda_i = 0$, $\lambda_i > 0$, there is a $\{V_{r'_i}\}$ such that $\lambda_i U_{r_i} \subseteqq V_{r'_i}$.

(R, L_3) Let g be any point in E. For any $\{U_{r_i}\}$ there is a $\{V_{r'_i}(g)\}$, which is a fundamental sequence of neighbourhoods with respect to g, such that $g + U_{r_i} \subseteqq V_{r'_i}(g)$ and conversely, for any $\{U_{r_i}(g)\}$ there is a $\{V_{r'_i}\}$ such that $U_{r_i}(g) \subseteqq g + V_{r'_i}$.

(E, 3) For any neighbourhood U_i and for any $r > 0$, there exists some U_j such that $rU_i \supset U_j$.

(E, 4) For any neighbourhood $U_i(g)$ with respect to any g and for any $U_j(g)$ with $U_j(g) \subset U_i(g)$ and $j > i$, if $f \in U_j(g)$ there exists some neighbourhood U_k such that $f + U_k \subset U_i(g)$.

Next, let a linear space F be a ranked space with indicator ω_0, which satisfies the following conditions.

(F, 1) This is the same as (E, 1).

(F, 2) This is the same as (E, 2).

(F, 3) For any neighbourhood U_i and for any $\{V_{\gamma_i}\}$, there exists some integer i_0 such that $U_i \supset V_{\gamma_j}$ if $j > i_0$.

(F, 4) For any neighbourhood U_i and for any $\alpha > 0$, if g does not belong to αU_i, then there exist some $\varepsilon = \varepsilon(U_i)$ (with $0 < \varepsilon < 1$) and some neighbourhood V_j such that

$$\alpha(1-\varepsilon)U_i \cap (V_j + g) = \phi.$$

Now, we can prove the following theorem.

Theorem. *Suppose E and F are the above-mentioned spaces. Let $\mathfrak{T}$ be a family of continuous linear operators from E into F. If for any $g \in E$, there are some fundamental sequence of neighbourhoods $\{U_{\gamma_i}\}$, and some $\beta_i > 0$ such that $\{Tg\}_{T \in \mathfrak{T}} \subset \beta_i U_{\gamma_i}$ for all i, then for every U_j in F, there exist some neighbourhood V_i in E, some $r > 0$ and some point $f \in E$ such that $U_j \supset \{Tg\}_{T \in \mathfrak{T}}$ for $g \in rV_i + f$.*

Proof. Assume the contrary and suppose that for a U_{j_0} and any $rV_i + f$ there exist some $g \in rV_i + f$ and some $T \in \mathfrak{T}$ such that $U_{j_0} \not\ni Tg$.

Now, let $V_1 + f_1$ be an arbitrary neighbourhood in E and α_1 be a real number such that $\alpha_1 > 1$. And suppose $V_{\gamma_1'} + f_1$ is the neighbourhood such that $1 < \gamma_1 < \gamma_1'$ and $V_1 \supset V_{\gamma_1} \supset V_{\gamma_1'}$. Then there exist some g_1 belonging $(1/\alpha_1)(V_{\gamma_1'} + f_1)$ and some $T_{n_1} \in \mathfrak{T}$ such that $T_{n_1} g_1 \notin U_{j_0}$.

Hence we have $T_{n_1} \alpha_1 g_1 \notin \alpha_1 U_{j_0}$ for $\alpha_1 g_1 \in V_{\gamma_1'} + f_1$. Following (F, 4), there exist a number $\varepsilon = \varepsilon(U_{j_0})$ with $0 < \varepsilon < 1$ and U_l such that

$$\alpha_1(1-\varepsilon)U_{j_0} \cap (T_{n_1}\alpha_1 g_1 + U_l) = \phi.$$

On the other hand, since T_{n_1} is continuous, to U_l in F there corresponds a neighbourhood V_{γ_2} in E such that

$$T_{n_1} g - T_{n_1}\alpha_1 g_1 \in U_l \quad \text{if} \quad g - \alpha_1 g_1 \in V_{\gamma_2}.$$

Consequently we have

$$T_{n_1} g \notin \alpha_1(1-\varepsilon)U_{j_0} \quad \text{for} \quad g \in \alpha_1 g_1 + V_{\gamma_2}.$$

By condition (E, 4) we can consider V_{γ_2} with property that

$$\alpha_1 g_1 + V_{\gamma_2} \subset f_1 + V_{\gamma_1} \subset f_1 + V_1.$$

Next, let α_2 be a real number such that $\alpha_2 > 2$ and suppose $V_{\gamma_3'} + \alpha_1 g_1$ is the neighbourhood such that $\gamma_2 < \gamma_3 < \gamma_3'$ and $V_{\gamma_2} \supset V_{\gamma_3} \supset V_{\gamma_3'}$.

Then there exist some g_2 belonging $(1/\alpha_2)(V_{\gamma_3'} + \alpha_1 g_1)$ and some $T_{n_2} \in \mathfrak{T}$ such that $T_{n_2} g_2 \notin U_{j_0}$. Hence we have $T_{n_2}\alpha_2 g_2 \notin \alpha_2 U_{j_0}$ for $\alpha_2 g_2 \in V_{\gamma_3'} + \alpha_1 g_1$. Following (F, 4), there exist a number $\varepsilon = \varepsilon(U_{j_0})$ with $0 < \varepsilon < 1$ and $U_{l'}$ such that

$$\alpha_2(1-\varepsilon)U_{j_0} \cap (T_{n_2}\alpha_2 g_2 + U_{l'}) = \phi.$$

On the other hand, since T_{n_2} is continuous, to $U_{l'}$ in F there corresponds a neighbourhood V_{γ_4} in E such that

$$T_{n_2}g - T_{n_2}\alpha_2 g_2 \in U_{l'} \quad \text{if} \quad g - \alpha_2 g_2 \in V_{r_4}.$$

Consequently we have $T_{n_2}g \notin \alpha_2(1-\varepsilon)U_{j_0}$ for $g \in \alpha_2 g_2 + V_{r_4}$.

By condition (E, 4) we can consider V_{r_4} with property that

$$\alpha_2 g_2 + V_{r_4} \subset \alpha_1 g_1 + V_{r_3} \subset \alpha_1 g_1 + V_{r_2}.$$

Repeating the foregoing argument, we have

$$V_1 + f_1 \supset V_{r_1} + f_1 \supset V_{r_2} + \alpha_1 g_1 \supset V_{r_3} + \alpha_1 g_1 \supset V_{r_4} + \alpha_2 g_2 \supset V_{r_5} + \alpha_2 g_2 \supset \cdots$$
$$\text{with } 1 < r_1 < r_2 < r_3 < r_4 < r_5 < \cdots$$

and

$$T_{n_i} g \notin \alpha_i(1-\varepsilon)U_{j_0} \quad \text{for} \quad g \in \alpha_i g_i + V_{r_{2i}}.$$

Since the sequence $\{\alpha_i g_i\}$ is a Cauchy sequence, it has a limiting element $g_0 \in E$. Hence we have $g_0 \in \alpha_i g_i + V_{r_{2i}}$ for all i.

Consequently $T_{n_i} g_0 \notin \alpha_i(1-\varepsilon)U_{j_0}$ for all i.

This is a contradiction to the hypotheses.

Corollary (Banach-Steinhaus theorem). *Suppose E is the above-mentioned space with the same property as* (F, 3).

Let F be the above-mentioned space with the following additional properties.

(F, 5) The neighbourhoods of the origin are symmetric (i.e. if $g \in U_i$, then $-g \in U_i$).

(F, 6) For any $g \in F$ and any U_i, there exists some $\alpha > 0$ such that $g \in \alpha U_i$.

(F, 7) For any $\lambda > 0$, $\mu > 0$ and any U_i, we have $\lambda U_i + \mu U_i \subset (\lambda + \mu) U_i$.

(F, 8) For given distincts g_1, g_2, there exists some U_i such that $(g_1 + U_i) \not\ni g_2$.

And let $\{T_n\}_{n=1,2,\ldots}$ be a sequence of continuous linear operators from E into F. If $Tg = \lim T_n g$ exists for any $g \in E$, then T is a continuous linear operator from E into F.

Proof. Let $\{U_{r_i}\}$ be an arbitrary fundamental sequence of neighbourhoods in F. By the foregoing theorem, for any $U_{r_i} \in \{U_{r_i}\}$ there exists some $r_i V_{r_i'} + f_i$ such that $T_n g \in U_{r_i}$ for all n if $g \in r_i V_{r_i'} + f_i$. On the other hand, since $\{T_n f_i\}_{n=1,2,\ldots}$ converges, for U_{r_i} there exists some $\alpha_i > 0$ such that $\{T_n f_i\}_{n=1,2,\ldots} \subset \alpha_i U_{r_i}$. Now, let $\{\delta_i\}$ be the sequence of real numbers such that $\delta_i > 0$, $\delta_i \downarrow 0$ and $\delta_i \alpha_i \downarrow 0$.

Suppose $g_j \to g_0$ in E, then for sufficiently large N and $j > N$, we have $g_j - g_0 \in \delta_i r_i V_{r_i'}$. Hence we obtain

$$T_n\left(\frac{g_j - g_0}{\delta_i} + f_i\right) \in U_{r_i}, \qquad \text{for all } n$$

and $T_n(g_j - g_0) + \delta_i T_n f_i \in \delta_i U_{r_i}$. Then we have

$$T(g_j - g_0) = (T - T_n)(g_j - g_0) + T_n(g_j - g_0) + \delta_i T_n f_i - \delta_i T_n f_i$$
$$\in (T - T_n)(g_j - g_0) + \delta_i U_{r_i} - \delta_i T_n f_i.$$

Since $\{T_n(g_j - g_0)\}_{n=1,2,\ldots}$ converges, for sufficiently large N' and $n > N'$ we have $(T - T_n)(g_j - g_0) \in U_{r_i}$.

Consequently we obtain

$$T(g_j-g_0)\in U_{r_i}+\delta_i U_{r_i}+\delta_i\alpha_i U_{r_i}, \qquad \text{for } j>N.$$

By the Washihara's conditions $(\mathrm{R},\mathrm{L}_2)'$ (2) and $(\mathrm{R},\mathrm{L}_1)$, there exists $\{W_{r_i'}\}$ such that $U_{r_i}+\delta_i U_{r_i}+\delta_i\alpha_i U_{r_i}\subset W_{r_i'}$, and $T(g_j-g_0)\in W_{r_i'}$.

Hence we assert that T is continuous.

We shall introduce a new axiom.

(E, 4)′ Given any neighbourhood $U_i(g)$, there exists some $U_j(g)$ (with $U_j(g)\subset U_i(g)$ and $j>i$) so that for any $f\in U_j(g)$ we have some U_k such that $f+U_k\subset U_i(g)$.

Then we can prove the above-mentioned theorem and corrollary in the space E having (E, 4)′ in place of (E, 4).

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References

[1] K. Kunugi: Sur les espaces complets et régulièrement complets. I. Proc. Japan Acad., **30**, 553–556 (1954).

[2] ——: Sur la méthode des espaces rangés. I, II. Proc. Japan Acad., **42**, 318–322, 549–554 (1966).

[3] M. Washihara: On ranked spaces and linearity. II. Proc. Japan Acad., **45**, 238–242 (1969).

[4] Y. Nagakura: On closed graph theorem. Proc. Japan Acad., **48**, 665–668 (1972).