70. *Finitely Generated $\mathfrak{N}$-Semigroup and Quotient Group*

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1. Introduction. An $\mathfrak{N}$-semigroup is a commutative cancellative archimedean semigroup which has no idempotent. The structure and construction of finitely generated or power joined $\mathfrak{N}$-semigroups were studied by [2], [3], [5], [6], and also by [4] from the more general point of view. This paper treats finitely generated $\mathfrak{N}$-semigroups as subsemigroups of the direct product of the positive integer semigroup and a finite abelian group by using the quotient group and its torsion subgroup. Finitely generated $\mathfrak{N}$-semigroups are characterized by their quotient group.

2. Preliminaries. In this paper we denote the additive semigroup of integers, positive integers, negative integers, non-negative integers, and positive rational numbers by Z, Z_+, Z_-, Z_+^0, and R respectively.

Proposition 1 ([1], [6]). *Let G be an abelian group and $I: G\times G \to Z_+^0$ be a function satisfying*

(1.1) $I(\alpha,\beta)=I(\beta,\alpha)$ *for all* $\alpha,\beta\in G$.

(1.2) $I(\alpha,\beta)+I(\alpha\beta,\gamma)=I(\alpha,\beta\gamma)+I(\beta,\gamma)$ *for all* $\alpha,\beta,\gamma\in G$.

(1.3) $I(\varepsilon,\alpha)=1$ (*ε being the identity of G*) *for all* $\alpha\in G$.

(1.4) *For each $\alpha\in G$ there is $m\in Z_+$ such that* $I(\alpha,\alpha^m)>0$.

Let $S=\{(x,\alpha): x\in Z_+^0, \alpha\in G\}$. Define an operation

$$(x,\alpha)(y,\beta)=(x+y+I(\alpha,\beta),\alpha\beta).$$

Then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup can be obtained in this manner.

S is denoted by $S=(G;I)$. The group G is termed the structure group of S with respect to $(0,\varepsilon)$, the function I is called an index function or $\mathcal{I}$-function corresponding to G. For a given $\mathfrak{N}$-semigroup S, for each $a\in S$, the relation ρ_a on S is defined by

$$x\rho_a y \text{ if and only if } a^m x=a^n y \text{ for some } m,n\in Z_+.$$

Then ρ_a is a congruence on S and $G_a=S/\rho_a$ is an abelian group. Each ρ_a-class contains exactly one element p_α, $\alpha\in G_a$, such that $p_\alpha\notin Sa$. Then S is isomorphic onto $(G_a;I_a)$ where $p_{\alpha\beta}=a^{I_a(\alpha,\beta)}p_\alpha p_\beta$.

A commutative semigroup S is called power joined if for every $a,b\in S$ there are $m,n\in Z_+$ such that $a^m=b^n$. If S is power joined, it is archimedean.

Proposition 2 ([5]). *An $\mathfrak{N}$-semigroup $S=(G;I)$ is power joined if* *and only if G is periodic. $S=(G;I)$ is finitely generated if and only if G is finite.*

Therefore a finitely generated $\mathfrak{N}$-semigroup is power joined. Let $S=(G;I)$ be a finitely generated $\mathfrak{N}$-semigroup. Define $\varphi: G \to R$ by

$$\varphi(\alpha)=\frac{1}{|G|}\sum_{\xi \in G} I(\alpha, \xi). \tag{3.0}$$

Proposition 3 ([5]). *The function φ satisfies the following conditions.*

(3.1) $\varphi(\varepsilon)=1$, *ε the identity element of G.*

(3.2) $\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)$ *is a non-negative integer for all $\alpha, \beta \in G$.*

(3.3) $I(\alpha, \beta)=\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)$ *for all $\alpha, \beta \in G$.*

If $\varphi: G \to R$ satisfies (3.1) *and* (3.2), *and if I is defined by* (3.3), *then I satisfies* (1.1) *through* (1.4).

In the sense of (3.0) and (3.3) there is a one-to-one correspondence between φ and I for a fixed G.

Proposition 4 ([2]). *S is a finitely generated $\mathfrak{N}$-semigroup if and only if S is isomorphic onto a subdirect product of a positive integer additive semigroup and a finite abelian group.*

Let S be a commutative and cancellative semigroup. Consider a congruence τ on $S\times S$ defined by $(x, y)\tau(z, u)$ if and only if $xu=yz$ in S. Then $(S\times S)/\tau$ is a group which contains a subsemigroup isomorphic to S. We term $(S\times S)/\tau$ the quotient group of S and denote this group by $Q(S)$.

Proposition 5 ([7]). *Let $S=(G;I)$ where G need not be finite. $Q(S)$ is the abelian extension of Z by G with respect to a factor system $f(\alpha, \beta)$ defined by*

$$f(\alpha, \beta)=I(\alpha, \beta)-1. \tag{5.1}$$

We denote $Q(S)$ by $Q(S)=\mathrm{ext}(Z, G; f)$, i.e.,

$$Q(S)=\{(m, \alpha): m \in Z, \alpha \in G\}$$

in which $(m, \alpha)(n, \beta)=(m+n+f(\alpha, \beta), \alpha\beta)$.

3. Structure and construction. By Proposition 4 or [4], S is a finitely generated $\mathfrak{N}$-semigroup if and only if S is a subsemigroup of $Z_+\times K$ for some finite abelian group K. The following theorem, however, characterizes finitely generated $\mathfrak{N}$-semigroups in terms of a refined condition (6.3) or their quotient group.

Theorem 6. *The following are equivalent.*

(6.1) *S is a finitely generated $\mathfrak{N}$-semigroup.*

(6.2) *S is an $\mathfrak{N}$-semigroup and $Q(S)\cong Z\times H$ for some finite abelian group H.*

(6.3) *S is a subsemigroup of $Z_+\times H$ such that $(Z_+\times H)\backslash S$ is finite.**)

Proof. Let $S=(G;I)=(G;\varphi)$ and $g=|G|>1$. (6.1)$\Rightarrow$(6.2). $Q(S)$

*) By $(Z_+\times H)\backslash S$ we mean $(Z_+\times H)-S$.

$=\text{ext}(Z, G; f)$ where G is a finite abelian group with $g=|G|>1$. First define $\varphi: G \to R$ by (3.0) and then define $\theta: Q(S)\to Z$ by

(6.4) $$\theta(m, \alpha)=g\cdot(m-1+\varphi(\alpha)), \qquad (m, \alpha)\in Q(S).$$

By using (5.1), (3.3), it is easy to show that θ is a homomorphism of $Q(S)$ into Z. Clearly $\theta(Q(S))\neq\{0\}$, so $\theta(Q(S))$ is isomorphic onto Z. Without loss of generality, $\theta: Q(S)\to Z$ is assumed to be surjective. In order that (m, α) be in the kernel of θ, it is necessary that $m-1+\varphi(\alpha)=0$, hence $\varphi(\alpha)$ has to be a positive integer. But, there is at most one such (m, α) for each $\alpha\in G$. Hence, the kernel of θ, denoted by H, is finite. Thus $Q(S)$ is homomorphic onto the free group Z. By the theorem in the abelian group theory, $Q(S)$ is isomorphic onto $Z\times H$.

(6.2)$\Rightarrow$(6.3). Suppose that $Q(S)=Z\times H$ identifying $Q(S)$ with $Z\times H$ and that $S\subset Q(S)$. Note that elements of S are denoted by $(m, \alpha), (n, \beta)$ but the operation is $(m, \alpha)(n, \beta)=(m+n, \alpha\beta)$. Then $Z\times H=(Z_-\times H)\cup(\{0\}\times H)\cup(Z_+\times H)$. We now prove that $S\cap(\{0\}\times H)=\emptyset$ and either $S\subseteq Z_+\times H$ or $S\subseteq Z_-\times H$. Suppose $(0, \alpha)\in S$. Then $(0, \alpha)^{|H|}=(0, \varepsilon)\in S$, a contradiction, since S cannot contain the identity $(0, \varepsilon)$ of $Z\times H$. Suppose that $(x, \alpha)\in Z_+\times H$ and $(y, \beta)\in Z_-\times H$ and both are in S. Then $(x, \alpha)^{-y}\cdot(y, \beta)^x=(0, \gamma)\in S$ for some $\gamma\in H$ where $-y\in Z_+$. This is contrary to the above result. As $Z_+\times H$ and $Z_-\times H$ are isomorphic, we can assume that $S\subseteq Z_+\times H$. To show that $(Z_+\times H)\backslash S$ is finite, we need only show, equivalently, that (i) for each $\alpha\in H$, there is a positive integer l such that $(l, \alpha)\in S$, (ii) there is $j\in Z_+$ such that $(k, \varepsilon)\in S$ for all $k\geq j$ where ε is the identity of H. To prove (i) we note that since $Z\times H$ is the quotient group of S, for each $\alpha\in H$ we can find (m, β), $(n, \delta)\in S$ such that $(m, \beta)(-n, \delta^{-1})=(0, \alpha)$. As H is finite, there is $-p\in Z_+$ such that $\delta^{-p}=\delta^{-1}$. But, then $(m, \beta)(n, \delta)^{-p}=(m-pn, \alpha)$ as we mentioned. To show (ii) we first note that P is a positive integer additive semigroup such that $|Z_+\backslash P|<\infty$ if and only if P contains two elements which are relatively prime. Now we prove (ii). By (i) $S\cap(Z_+\times\{\varepsilon\})\neq\emptyset$ and it is isomorphic to a subsemigroup of Z_+. We show that (n, ε) and $(n+1, \varepsilon)$ are in S for some $n\in Z_+$. Since $Z\times H$ is the quotient group of S, we can find $(p, \alpha), (q, \beta)\in S$ such that $(p, \alpha)(q, \beta)^{-1}=(1, \varepsilon)$, whence $p=q+1$ and $\alpha=\beta$. Thus (q, α) and $(q+1, \alpha)$ are in S. We may assume $\alpha\neq\varepsilon$. Let r be the order of α in H. Then $r>1$. Clearly $(q, \alpha)^{r-1}(q+1, \alpha)$ and $(q, \alpha)^r$ are in S, but

$$(q, \alpha)^{r-1}(q+1, \alpha)=(qr+1, \varepsilon), \qquad (q, \alpha)^r=(qr, \varepsilon).$$

Thus we have found $n=qr$.

(6.3)$\Rightarrow$(6.1). It is immediate to show that any subsemigroup of $Z_+\times H$ is an $\mathfrak{N}$-semigroup. Archimedeaness of S follows from power joinedness of S. We show only that S is finitely generated. It is well known that any positive integer additive semigroup is finitely generated, therefore $S\cap(Z_+\times\{\varepsilon\})$ is generated by a finite subset A. Since $(Z_+\times H)\setminus S$ is finite, for each $\alpha\in H$, there is a smallest $k_\alpha\in Z_+$ such that $(x,\alpha)\in S$ for all $x\geq k_\alpha$. Let $l_\alpha=k_\varepsilon+k_\alpha$. It can be easily shown that S is generated by a subset of the set

$$A\cup\{(x,\alpha):\varepsilon\neq\alpha\in H, x<l_\alpha\}.$$

Hence S is finitely generated. This completes the proof. Q.E.D.

Let $H'=\{\alpha\in G:\varphi(\alpha)\in Z_+\}$. Then $H=\{(1-\varphi(\alpha),\alpha):\alpha\in H'\}$. H is the torsion subgroup of $Z\times H$ and $H\cong H'$. The embedding $S\to Z_+\times H$ is a universal repelling object in the category of the embeddings of S into finitely generated steady $\mathfrak{N}$-semigroups. (See [8].) As a consequence of Theorem 6 we get immediately the following theorem.

Theorem 7. *Let H be a finite abelian group, and $\mathcal{A}$ be a mapping of H into the power set 2^{Z_+} of Z_+, denoted by $\alpha\mapsto\mathcal{A}(\alpha)$, which satisfies*
(7.1) $\mathcal{A}(\alpha)+\mathcal{A}(\beta)\subseteq\mathcal{A}(\alpha\beta)$ *for all* $\alpha,\beta\in H$,
(7.2) $|Z_+\setminus\mathcal{A}(\varepsilon)|<\infty$.
Let $S=\{(x,\alpha):x\in\mathcal{A}(\alpha),\alpha\in H\}$ in which a binary operation is defined by $(x,\alpha)\cdot(y,\beta)=(x+y,\alpha\beta)$. Then S is a finitely generated $\mathfrak{N}$-semigroup whose quotient group is $Z\times H$. All finitely generated $\mathfrak{N}$-semigroups can be obtained in this manner.

Thus a finitely generated $\mathfrak{N}$-semigroup S is determined by a finite abelian group H and a map $\mathcal{A}:H\to 2^{Z_+}$; so S is denoted by

$$S=(H,\mathcal{A}).$$

Theorem 8. *Let $S=(H,\mathcal{A})$ and $T=(K,\mathcal{B})$. Then $S\cong T$ if and only if*
(8.1) *there is an isomorphism f of H onto K and*
(8.2) *there is an element $\sigma\in K$ such that for each $\alpha\in K$,*

$$\mathcal{B}(\alpha)=\{x\in Z_+:\sigma^x f(\xi)=\alpha \text{ and } x\in\mathcal{A}(\xi) \text{ for some } \xi\in H\}.$$

Proof. Assume that $S\cong T$. It is routine to prove that any isomorphism of S onto T can be uniquely extended to an isomorphism of $Q(S)$ onto $Q(T)$. By Theorem 6, $Q(S)\cong Z\times H$ and $Q(T)\cong Z\times K$. Hence $S\cong T$ implies $Z\times H\cong Z\times K$. Let h be an isomorphism of $Z\times H$ onto $Z\times K$, and let (x,α) and $[y,\beta]$ denote elements of $Z\times H$ and $Z\times K$ respectively. As H and K are the torsion subgroups of $Z\times H$ and $Z\times K$ respectively, h induces an isomorphism H onto K, denote $f=h|_H$, i.e., $h(0,\xi)=[0,f(\xi)]$. Now let $[l,\sigma]=h(1,\varepsilon)$. Then we have

$$\begin{aligned}h(x,\xi)&=h((1,\varepsilon)^x(0,\xi))=(h(1,\varepsilon))^x h(0,\xi)\\&=[l,\sigma]^x[0,f(\xi)]=[lx,\sigma^x f(\xi)].\end{aligned}$$

In order that h be onto, l has to be 1 or -1. Without loss of generality we assume that $S\subseteq Z_+\times H$ and $T\subseteq Z_+\times K$. Therefore $l=1$. Thus we get

$$h(x,\xi)=[x,\sigma^x f(\xi)]. \tag{8.3}$$

It is easy to see that h defined by (8.3) is an isomorphism of $Z\times H$ onto

$Z \times K$. Every isomorphism of S onto T is the restriction of some h given by (8.3). Now the theorem is an immediate consequence.

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