93. *Amenable Transformation Groups. II*

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Introduction. Let X be a nonvoid set and G be a group of transformations of X onto itself. Then we shall say the pair $X=(G, X)$ is a *transformation group*. Let $m(X)$ be the Banach space of all bounded real functions on X and $m(X)^*$ the conjugate Banach space of $m(X)$. For every $s \in G$, define the mapping $l_s: m(X) \to m(X)$ by $l_s f = {}_s f$ for any $f \in m(X)$ where ${}_s f(x) = f(sx)$ for $x \in X$, and denote by L_s the adjoint of l_s. For $\varphi \in m(X)^*$ it is called a *mean* if $\varphi \geqq 0$ and $\varphi(I_X)=1$ where I_X is the constant one function on X. A mean φ is called *multiplicative* if $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$ for any $f, g \in m(X)$. For a subset K of G, a mean φ is *K-invariant* if $L_s \varphi = \varphi$ for all $s \in K$. We denote by δ_x the Dirac measure at $x \in X$. Let $IM(X)$ $[\beta X]$ be the set of all G-invariant [multiplicative] means. We shall say the transformation group $X=(G, X)$ is *amenable* if $IM(X)$ is nonempty.

The purpose of this paper is to characterize the transformation group $X=(G, X)$ such that $IM(X) \cap Co(\beta X)$ is nonempty where $Co(\beta X)$ is the convex hull of βX and to study the extreme point of the convex set $IM(X) \cap Co(\beta X)$. For semigroups the analogous problem is investigated by A. T. Lau in [3] and [4].

§1. Multiplicative means. In this section we give the Lemmas used in later sections. Let $X=(G, X)$ be a transformation group and $\varphi \in m(X)^*$ be a mean. For any subset A of X, we write $\varphi(A)$ instead of $\varphi(I_A)$ where I_A is the characteristic function of A. We put $H(\varphi) = \{s \in G : L_s \varphi = \varphi\}$.

Lemma 1. *Let $\Phi = \{\varphi_i \in \beta X : i=1, 2, \cdots, m$ and $\varphi_i \neq \varphi_j$ if $i \neq j\}$ and $\Psi = \{\psi_i \in \beta X : i=1, 2, \cdots, n$ and $\psi_i \neq \psi_j$ if $i \neq j\}$. If $\sum_{i=1}^m \lambda_i \varphi_i = \sum_{i=1}^n \mu_i \psi_i$ where λ_i's and μ_i's are positive numbers, then $\Phi = \Psi$.*

Lemma 2. *Let $\varphi_0 \in \beta X$. For a subset $\{a_1, a_2, \cdots, a_n\}$ of G put $\varphi_i = L_{a_i} \varphi_0 \in \beta X$ for $1 \leqq i \leqq n$. If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are mutually distinct, there is a subset $A_0 \subset X$ such that for any $1 \leqq i, j \leqq n$ $\varphi_i(A_j) = \delta_{ij}$ and $A_i \cap A_j = \phi$ $(i \neq j)$ where $A_i = a_i A_0$.*

Now for a mean φ we consider the condition (♯): *there is a positive constant c such that $\varphi(A) \geqq c$ or $\varphi(A)=0$ for any $A \subset X$.* If the condition (♯) is satisfied, there is a subset $A \subset X$ such that $\varphi(A) > 0$ and that $\varphi(A \cap B)$ is equal to $\varphi(A)$ or 0 for any $B \subset X$. For example, every $\varphi \in Co(\beta X)$ satisfies the condition (♯).

Lemma 3. *Let $\varphi \in IM(X)$ satisfy the condition (♯) and A be a* *subset of X such that $\varphi(A)>0$ and that $\varphi(A\cap B)$ is equal to $\varphi(A)$ or 0 for any $B\subset X$. Putting $H=\{s\in G:\varphi(sA\cap A)=\varphi(A)\}$ and $\varphi_A(g)=\varphi(I_A\cdot g)/\varphi(A)$ for any $g\in m(X)$, we have the following:*

(1) *H is a subgroup of G with finite index.*

(2) *For any $B,C\subset X$ and $s\in H$, $\varphi(A\cap B\cap C)=\varphi(sA\cap B\cap C)=\varphi(A\cap sB\cap C)$.*

(3) *For any $f,g\in m(X)$ and $s\in H$, $\varphi_A(f\cdot {}_sg)=\varphi_A(f\cdot g)$.*

(4) *$H=\{s\in G:\varphi_A(f\cdot {}_sg)=\varphi_A(f\cdot g)$ for any $f,g\in m(X)\}=H(\varphi_A)$.*

§2. Main theorem. In this section we give various characterizations of a transformation group $X=(G,X)$ with G-invariant mean in the convex hull of βX. For any finite set M denote by $|M|$ the cardinality of M.

Theorem 1. *The following conditions on a transformation group $X=(G,X)$ are equivalent:*

(1) *$IM(X)\cap Co(\beta X)$ is nonempty.*

(2) *There is $\varphi\in IM(X)$ such that the subgroup $H(\varphi)$ of G has finite index.*

(3) *There is an integer $n\geqq 1$ such that for any finite subset K of G there exists a finite subset F_K of X having the properties $|F_K|=n$ and $sF_K=F_K$ for all $s\in K$.*

(4) *For some integer $n\geqq 1$ there is a net $\{p^\alpha=1/n\sum_{i=1}^n\delta_{x_i^\alpha}\}$ in $Co(\beta X)$ such that $\lim_\alpha\|L_sp^\alpha-p^\alpha\|=0$ for any $s\in G$.*

Proof. (1)⇒(2): Let $\varphi=\sum_{i=1}^n\lambda_i\varphi_i\in IM(X)\cap Co(\beta X)$ where φ_i's are mutually distinct elements in βX. Then, by the G-invariancy of φ and Lemma 1, we have $\{\varphi_1,\varphi_2,\cdots,\varphi_n\}=\{L_s\varphi_1,L_s\varphi_2,\cdots,L_s\varphi_n\}$ for all $s\in G$. So each $H(\varphi_i)$ has finite index in G.

(2)⇒(3): For $\varphi\in\beta X$ assume that $H(\varphi)$ has finite index in G. Let $\{a_1=e,a_2,\cdots,a_n\}$ be a representative system of the left coset space $G/H(\varphi)$ and put $\varphi_i=L_{a_i}\varphi$ for any $1\leqq i\leqq n$. Then, by Lemma 2, there is a subset $A\subset X$ such that for any $1\leqq i,j\leqq n$ $\varphi_i(A_j)=\delta_{ij}$ and $A_i\cap A_j=\phi$ $(i\neq j)$ where $A_i=a_iA$. For any $1\leqq i\leqq n$ and $s\in G$ there correspond an integer k and $h_{si}\in H(\varphi)$ such that $sa_i=a_kh_{si}$. Now for any finite subset K of G put $H_K=\{h_{si}:s\in K$ and $i=1,2,\cdots,n\}$. Since $\varphi=\varphi_1$ is a multiplicative $H(\varphi)$-invariant mean, by Theorem 3 in [5], there is $x\in A_1$ such that $hx=x$ for all $h\in H_K$. Putting $F_K=\{a_1x,a_2x,\cdots,a_nx\}$, clearly we have $|F_K|=n$ and $sF_K=F_K$ for all $s\in K$.

The other implications (3)⇒(4)⇒(1) are obtained by the same way as in A. T. Lau [2, Theorems 5.3 and 5.5]. q.e.d.

Similarly we have

Theorem 2. *Let $X=(G,X)$ be a transformation group and n be a fixed positive integer. Then the following conditions are equivalent:*

(1) *There is a G-invariant mean φ of the form $\varphi=\sum_{i=1}^n\lambda_i\varphi_i$ where*

for any $1\leqq i\leqq n$ $\varphi_i\in\beta X$ *and* $\lambda_i>0$, $\sum_{i=1}^n\lambda_i=1$ *and* $\varphi_i\neq\varphi_j$ *if* $i\neq j$.

(2) *There are mutually disjoint subsets* $A_1, A_2, \cdots, A_n$ *of* X *such that for any finite subset* K *of* G *there exists a finite subset* $F_K=\{x_1, x_2, \cdots, x_n: x_i\in A_i$ *for any* $1\leqq i\leqq n\}$ *with the property* $sF_K=F_K$ *for all* $s\in K$.

§3. Extreme points of $IM(X)\cap Co(\beta X)$. Let $X=(G,X)$ be an amenable transformation group such that $IM(X)\cap Co(\beta X)$ is nonempty. Each extreme point of the convex set $IM(X)\cap Co(\beta X)$ is also an extreme point of $IM(X)$. For $\varphi\in\beta X$ assume that $H(\varphi)$ has finite index in G. Let $\{a_1=e, a_2, \cdots, a_n\}$ be a representative system of $G/H(\varphi)$ and put $\varphi_i=L_{a_i}\varphi$ for any $1\leqq i\leqq n$. Then $\psi=1/n\sum_{i=1}^n\varphi_i$ is an extreme point of $IM(X)\cap Co(\beta X)$. In this case $\tilde{H}=\bigcap_{i=1}^n H(\varphi_i)$ has finite index in G and each φ_i is $\tilde{H}$-invariant. Moreover, using Lemma 3, we can conclude that $\tilde{H}$ is equal to $\{s\in G: \psi(f\cdot{}_sg)=\psi(f\cdot g)$ for any $f, g\in m(X)\}$.

Conversely every extreme point of $IM(X)\cap Co(\beta X)$ is given in the above form.

Theorem 3. *Let $X=(G,X)$ be an amenable transformation group and φ be an extreme point of the convex set $IM(X)$. If $H=\{s\in G: \varphi(f\cdot{}_sg)=\varphi(f\cdot g)$ for any $f, g\in m(X)\}$ has finite index in G, then φ is in $IM(X)\cap Co(\beta X)$.*

Proof. Let $\{a_1=e, a_2, \cdots, a_n\}$ be a representative system of G/H and fix an arbitrary $f\in m(X)$ with $0\leqq f\leqq 1$. Now define $\nu\in m(X)^*$ as follows:

$$\nu(g)=\varphi(f)\cdot\varphi(g)-\frac{1}{n}\sum_{i=1}^n\varphi(f\cdot{}_{a_i}g)$$

for any $g\in m(X)$. Then $\nu(I_X)=0$ and $L_s\nu=\nu$ for any $s\in G$. Put $\varphi^\pm=\varphi\pm\nu$. Then we have easily $\varphi^\pm\in IM(X)$ and $\varphi=(\varphi^++\varphi^-)/2$. Since φ is extreme, we have $\nu\equiv 0$. Consequently we have for any $f, g\in m(X)$ with $0\leqq f\leqq 1$

$$\varphi(f)\cdot\varphi(g)=\frac{1}{n}\sum_{i=1}^n\varphi(f\cdot{}_{a_i}g). \tag{\#\#}$$

By the linearlity of φ, (##) is also valid for any $f, g\in m(X)$. For any $A\subset X$, by (##), it holds $\varphi(A)^2=1/n\sum_{i=1}^n\varphi(A\cap a_i^{-1}A)\geqq(1/n)\varphi(A)$. So $\varphi(A)=0$ or $\varphi(A)\geqq 1/n$. Thus φ satisfies the condition (#) in §1. Let $A\subset X$ have the properties that $\varphi(A)>0$ and that $\varphi(A\cap B)$ is equal to $\varphi(A)$ or 0 for any $B\subset X$. Then the subgroup $H_A=\{s\in G: \varphi(sA\cap A)=\varphi(A)\}$ contains H. Let $\{b_1=e, b_2, \cdots, b_m\}[\{c_1, c_2, \cdots, c_k\}]$ be a representative system of $G/H_A[H_A/H]$. Clearly it holds $\varphi(b_iA\cap b_jA)=\delta_{ij}\varphi(A)$ and $\varphi(b_ic_jA\cap A)=\delta_{i1}\varphi(A)$ for any $1\leqq i, j\leqq n$. Since $\{b_ic_j: i=1,2,\cdots,m$ and $j=1,2,\cdots,k\}$ is a representative system of G/H, by (##), we have:

$$\varphi(A)^2=\frac{1}{n}\sum_{i=1}^m\sum_{j=1}^k\varphi(b_ic_jA\cap A)=\frac{k}{n}\varphi(A)=\frac{1}{m}\varphi(A),$$

$$\varphi(A)\cdot\varphi(g)=\frac{1}{n}\sum_{i=1}^{m}\sum_{j=1}^{n}\varphi(I_A\cdot{}_{b_ic_j}g)=\frac{1}{m}\sum_{i=1}^{m}\varphi(I_{b_iA}\cdot g)$$

for any $g\in m(X)$. For $1\leqq i\leqq m$, put $A_i=b_iA$ and $\varphi_i(g)=\varphi(I_{A_i}\cdot g)/\varphi(A)$ $=m\varphi(I_A\cdot{}_{b_i}g)$ for any $g\in m(X)$. Then each $\varphi_i=L_{b_i}\varphi_1$ is an H-invariant mean and we have $\varphi=1/m\sum_{i=1}^{m}\varphi_i$. It remains to prove that each φ_i is multiplicative. Now again using the relation (##) we have

$$\varphi(I_A\cdot f)\cdot\varphi(I_A\cdot g)=(1/m)\varphi(I_A\cdot f\cdot g)$$

for any $f,g\in m(X)$. So $\varphi_1(f)\cdot\varphi_1(g)=m^2\varphi(I_A\cdot f)\varphi(I_A\cdot g)=m\varphi(I_A\cdot f\cdot g)$ $=\varphi_1(f\cdot g)$, that is, φ_1 is multiplicative. Consequently each φ_i is also multiplicative. q.e.d.

The following is a sufficient condition in order that every extreme point of $IM(X)$ is contained in $Co(\beta X)$, which is a generalization of Theorem 4 in [5].

Theorem 4. *Let $X=(G,X)$ be an amenable transformation group and H a subgroup of G with finite index. Then the following conditions are equivalent*:

(1) *For every $\varphi\in IM(X), f,g\in m(X)$ and $s\in H$ we have $\varphi(f\cdot{}_sg)=\varphi(f\cdot g)$.*

(2) *Every extreme point of $IM(X)$ is contained in the convex hull of all the H-invariant multiplicative means.*

(3) *Let $A\subset X$ have the property that $\varphi(A)>0$ for some $\varphi\in IM(X)$. Then, for any $s\in H$, there is $x\in A$ such that $sx=x$.*

References

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