92. *Extremely Amenable Transformation Semigroups*

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(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1973)

Introduction. Let X be a nonvoid set and S a semigroup of mappings of X into itself. Then we shall say the pair $\boldsymbol{X}=(S,X)$ is a *transformation semigroup*. Let $m(X)$ be the Banach space of all bounded real functions on X with the supremum norm and $m(X)^*$ the conjugate Banach space of $m(X)$. For $s\in S$, $l_s: m(X)\to m(X)$ be the linear mapping defined by $(l_sf)(x)=f(sx)$ for any $f\in m(X)$ and $x\in X$. An element $\varphi\in m(X)^*$ is called a *mean* on $m(X)$ if $\varphi\geqq 0$ and $\varphi(I_X)=1$ where I_X is the constant one function on X. A mean φ is called *invariant* if $\varphi(l_sf)=\varphi(f)$ for any $f\in m(X)$ and $s\in S$. A mean φ is *multiplicative* if $\varphi(f\cdot g)=\varphi(f)\cdot\varphi(g)$ for any $f,g\in m(X)$ where $f\cdot g$ is the pointwise product of f and g. Let $IM(\boldsymbol{X})$ [$MIM(\boldsymbol{X})$] be the set of all invariant [multiplicative invariant] means on $m(X)$. A transformation semigroup $\boldsymbol{X}$ is called *amenable* [*extremely amenable*] if $IM(\boldsymbol{X})$ [$MIM(\boldsymbol{X})$] is nonempty.

For any $s\in S$, put $A_s=\{x\in X: sx=x\}$. If $\cap\{A_s: s\in S\}$ is nonempty and x is in it, then the point measure δ_x(i.e., $\delta_x(f)=f(x)$ for any $f\in m(X)$) is a multiplicative invariant mean. In this case $\boldsymbol{X}$ is extremely amenable.

The purpose of this paper is to give various characterizations of extremely amenable transformation semigroup and to study the amenable one for which every extreme point in the set of all invariant means is multiplicative.

Now S be an abstract semigroup. Then, associating for any $s\in S$ the left translation $t\to st$ for $t\in S$, $\boldsymbol{S}_L=(S,S)$ can be regarded as a transformation semigroup. A semigroup S is called *extremely left amenable* if $\boldsymbol{S}_L$ is extremely amenable in the above sense. On extremely amenable semigroups it is investigated in detail by E. Granirer in [2], [3] and [4], and by T. Mitchell in [6]. Some results in [2] and [3] are contained in this paper as special cases.

§1. Finitely additive measure defined by a multiplicative mean. Let $\boldsymbol{X}=(S,X)$ be a transformation semigroup and $\varphi\in m(X)^*$ be a mean. For any $A\subset X$, we write $\varphi(A)$ instead of $\varphi(I_A)$ where I_A is the characteristic function of A. Then, since $\varphi(X)=1$, $A\to\varphi(A)$ is a finitely additive probability measure defined on the family of all the subsets of X. In what follows, let $s\in S$ be arbitrarily fixed and φ be a multiplicative mean such that $l_s^*\varphi=\varphi$. Then, for any $A\subset X$, $\varphi(A)$ is equal to 1 or 0 and $\varphi(sA)\geqq\varphi(A)$. Moreover we have $\varphi(A)=0$ if $sA\cap A$ is empty. For any integers $0\leqq i<k$, we define the subsets X^∞ and X_i^k of X as follow:

$X^\infty=\{x\in X: s^m x\neq s^n x$ for any distinct nonnegative integers $m, n\}$,

$X_i^k=\{x\in X: x, sx, \cdots, s^{k-1}x$ are mutually distinct and $s^k x=s^i x\}$.

Then $X=\bigcup_{k=1}^\infty(\bigcup_{i=0}^{k-1}X_i^k)\cup X^\infty$ is a partition of X and we have $sX^\infty\subseteq X^\infty$, $sX_0^k\subseteq X_0^k$ for any $k\geqq 1$, and $sX_i^k\subseteq X_{i-1}^{k-1}$ for any $1\leqq i<k$.

The following Lemma is obtained by slight modifications of Lemmas 1, 2 in [2].

Lemma 1. (1) *X^∞ is decomposed into the disjoint subsets X_1^∞ and X_2^∞ such that $sX_1^\infty\subseteq X_2^\infty$ and $sX_2^\infty\subseteq X_1^\infty$.*

(2) *For and $k\geqq 2$, X_0^k is decomposed into the mutually disjoint subsets $Y_1, Y_2, \cdots, Y_k$ such that $sY_k\subseteq Y_1$ and $sY_j\subseteq Y_{j+1}$ for $1\leqq j\leqq k-1$.*

Theorem 1. *Let φ be a multiplicative mean such that $l_s^*\varphi=\varphi$. Then we have $\varphi(A_s)=1$. So A_s is nonempty.*

Proof. From the fact mentioned in the above and Lemma 1 it follows that $\varphi(X^\infty)=0$, $\varphi(X_0^k)=0$ for any $k\geqq 2$ and $\varphi(X_i^k)\leqq(X_0^{k-i})$ for $1\leqq i<k$. Thus we have $\varphi(X_i^k)=0$ for $k-i\geqq 2$. Now suppose that $\varphi(A_s)=\varphi(X_0^1)=0$. Then $\varphi(X_{k-1}^k)=0$ for all $k\geqq 1$. So $\varphi\equiv 0$. This is a contradiction. q.e.d.

§ 2. Extremely amenability of transformation semigroups. Let $X=(S, X)$ be a transformation semigroup. A subset Y of X is said to be an *(F)-set* if $Y\cap A_s$ is nonempty for all $s\in S$. Denote by $\mathfrak{H}(X)$ the set of all functions h having the form $h=\sum_{i=1}^n f_i(g_i-l_{s_i}g_i)$ for some $f_1, \cdots, f_n, g_1, \cdots, g_n\in m(X)$ and $s_1, \cdots, s_n\in S$.

Theorem 2. *The following conditions on a transformation semigroup $X=(S, X)$ are equivalent:*

(0) *X is extremely amenable.*

(1) *For every finite subset K of S there is some $x\in X$ such that $sx=x$ for all $s\in K$.*

(2) *There is a net of point measures $\{\delta_{x_\alpha}: x_\alpha\in X\}$ in $m(X)^*$ such that $\lim_\alpha\|l_s^*\delta_{x_\alpha}-\delta_{x_\alpha}\|=0$ for any $s\in S$.*

(3) *For any finite partition $\{X_i: i=1, 2, \cdots, n\}$ of X, at least one of X_i's is an (F)-set.*

(4) *For each function h in $\mathfrak{H}(X)$ there is some $x\in X$ such that $h(x)=0$.*

Proof. (0)⇒(1): Let $\varphi\in MIM(X)$. The family $\mathfrak{A}=\{A\subset X: \varphi(A)=1\}$ has the finite intersection property. By Theorem 1, $\tilde{\mathfrak{A}}=\{A_s: s\in S\}$ is a subfamily of $\mathfrak{A}$. Therefore $\tilde{\mathfrak{A}}$ has also the finite intersection property.

(1)⇒(2): Let Δ be the family of all finite subsets of S ordered upwards by inclusion. Then Δ is a directed set. For each $\alpha\in\Delta$, by the condition (1), we can choose $x_\alpha \in X$ such that $sx_\alpha = x_\alpha$ for all $s \in \alpha$. Clearly the net $\{\delta_{x_\alpha} : \alpha \in \Delta\}$ satisfies the condition (2).

(2)⇒(0): Let $\{\delta_{x_\alpha}\}$ be the net satisfying the condition (2). Since the set of all means is w^*-compact in $m(X)^*$, this net has at least one w^*-cluster point φ. This φ is in $MIM(X)$.

(0)⇒(3): Let $\varphi \in MIM(X)$ and $\{X_i : i=1, \cdots, n\}$ a partition of X. Then $\varphi(X) = \sum_{i=1}^n \varphi(X_i) = 1$. So there is some X_i such that $\varphi(X_i)=1$. This X_i is an (F)-set.

(3)⇒(1): For every $s \in S$, by the condition (3), A_s is an (F)-set. For any finite subset $\{s_1, s_2, \cdots, s_n\}$ $(n \geqq 2)$ of S, put $X_n = \bigcap_{i=1}^n A_{s_i}$ and $X_{n-1} = \bigcap_{i=1}^{n-1} A_{s_i}$. Then $X = X_n \cup (X_{n-1} - A_{s_n}) \cup (A_{s_n} - X_{n-1}) \cup (A'_{s_n} \cap X'_{n-1})$ is a partition of X. $A_{s_n} - X_{n-1}$ is decomposed into the mutually disjoint subsets $B_j (j=1,2,\cdots,n-1)$ such that $B_j \cap A_{s_j} = \phi$ for $1 \leqq j \leqq n-1$. So, by the condition (3), X_n is an (F)-set. Therefore $\tilde{\mathfrak{A}}$ has the finite intersection property.

(1)⇒(4): This is obvious.

(4)⇒(0): This follows from Lemma 3(b) in [3]. q.e.d.

Similary we have

Theorem 3. *Let $X=(S,X)$ be a transformation semigroup and A a sebset of X. Then the following conditions are equivalent:*

(1) *There is $\varphi \in MIM(X)$ such that $\varphi(A)=1$.*

(2) *For every finite subset K of S there exists $x \in A$ such that $sx = x$ for all $s \in K$.*

§3. Amenable transformation semigroup for which each extreme point in $IM(X)$ is multiplicative. For an amenable transformation semigroup (S,X), a subset A of X is said to be a *(P)-set* if there is some $\varphi \in IM(X)$ such that $\varphi(A) > 0$.

Theorem 4. *The following conditions on an amenable transformation semigroup $X=(S,X)$ are equivalent:*

(1) *Each extreme point of the convex set $IM(X)$ is multiplicative.*

(2) *Every (P)-set of X is an (F)-set.*

(3) *For any $\varphi \in IM(X)$, $f, g \in m(X)$ and $s \in S$, we have $\varphi(f \cdot l_s g) = \varphi(f \cdot g)$.*

Proof. (1)⇒(2): Let A be a subset of X such that $\varphi(A) > 0$ for some $\varphi \in IM(X)$. Then, in this case, by Krein-Milman theorem ([5, p. 460]), there is some $\varphi_0 \in MIM(X)$ such that $\varphi_0(A)=1$. Clearly A is an (F)-set.

(2)⇒(3): Suppose that for some $\varphi \in IM(X)$ there exist $f, g \in m(X)$ and $s \in S$ such that $\varphi(f(l_s g - g)) > 0$. We put $A = \{x \in X : f(x)(g(sx) - g(x)) > 0\}$. Then $\varphi(A) > 0$ and $A \cap A_s = \phi$. This contradicts the condition (2).

(3)⇒(1): It is the same as the proof of Theorem 6 in [3]. q.e.d.

Corollary. *Let $X=(S,X)$ be a transformation semigroup such that S is extremely left amenable. Then every extremely point of $IM(X)$ is multiplicative.*

Proof. Note that in this case X is extremely. Now let A be a (P)-set and $\varphi(A)>0$ for some $\varphi \in IM(X)$. For any $s \in S$, by Theorem 2(1), there is some $t \in S$ such that $st=t$. Then, by the invariancy of φ, we have $\varphi(t^{-1}A)=\varphi(A)>0$ where $t^{-1}A=\{x \in X: tx \in A\}$. For any $x \in t^{-1}A$, we have $stx=tx$. Thus $tx \in A \cap A_s$. Therefore A is an (F)-set. From Theorem 4(2) it follows the corollary. q.e.d.

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