84. On Infinitesimal Automorphisms and Homogeneous Siegel Domains over Circular Cones

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Let $D(V,F)$ be a homogeneous Siegel domain of type I or type II, where V is a convex cone in a real vector space R and F is a V-hermitian form on a complex vector space W. Let $C(n)$ be the *circular cone* of dimension n ($n\geq 3$), that is, the set $\{(x_1,\cdots,x_n)\in \boldsymbol{R}^n\,;\, x_1>0, x_1x_2-x_3^2-,\cdots,-x_n^2>0\}$. In this note we will state a result on infinitesimal automorphisms of $D(V,F)$ and a method of constructing all homogeneous Siegel domains over circular cones. As an application, we will give the explicit form of a Siegel domain which is isomorphic to the exceptional bounded symmetric domain in $\boldsymbol{C}^{16}$ (; no explicit description of this Siegel domain has ever been obtained, as far as we know). The detailed results with their complete proofs will appear elsewhere.

1. Let $\mathfrak{g}_h$ (resp. $\mathfrak{g}_a$) denote the Lie algebra of all infinitesimal holomorphic (resp. affine) automorphisms of $D(V,F)$. Let $(z_1,\cdots,z_n, w_1,\cdots,w_m)$ be a canonical complex coordinate system of $R^C\times W$, where R^C is the complexification of R, $n=\dim_C R^C$, $m=\dim_C W$ and put $\partial=\sum_{1\leq k\leq n} z_k\partial/\partial z_k+1/2\sum_{1\leq \alpha\leq m} w_\alpha\partial/\partial w_\alpha$. Then the following results are known in [5], [10].

(1) *$\mathfrak{g}_h=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1$ is a graded Lie algebra and $\mathfrak{g}_a=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0$, where $\mathfrak{g}_\lambda$ ($\lambda=0, \pm 1/2, \pm 1$) is the λ-eigenspace of* ad (∂). *Furthermore $\mathfrak{g}_{-1}$ is identified with R as vector spaces.*

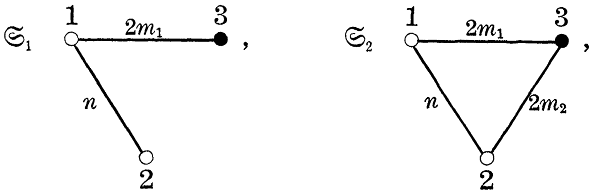
Considering (1) we denote by ρ the adjoint representation of the subalgebra $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}=R$, and we know $\rho(\mathfrak{g}_0)\subset\mathfrak{g}(V)\subset\mathfrak{gl}(R)$, where $\mathfrak{g}(V)$ denotes the Lie algebra of $\mathrm{Aut}(V)=\{g\in GL(R)\,;\, g(V)=V\}$. Using the descriptions of $\mathfrak{g}_{1/2}$, $\mathfrak{g}_1$ in terms of polynomial vector fields [7] and using the structure of the radical of $\mathfrak{g}_h$ [5] and the criterion of irreducibility of $D(V,F)$ [2], we get

Theorem 1. *If ρ is irreducible, then $\mathfrak{g}_h$ is simple or $\mathfrak{g}_h=\mathfrak{g}_a$.*

A homogeneous Siegel domain $D(V,F)$ of type II is said to be *non-degenerate* if the linear closure of $\{F(u,u)\,;\, u\in W\}$ in R coincides with R (cf. [3]).

Remark. Without the assumption of irreducibility of ρ, we can prove that $\mathfrak{g}_h = \mathfrak{g}_a$ if $D(V, F)$ is non-degenerate and $\mathfrak{g}_{1/2} = (0)$.

2. It is known in [4] that to each homogeneous Siegel domain of type II there corresponds a certain skeleton of type II. In view of facts in [4], [9], we can see that to each homogeneous Siegel domain $D(C(n+2), F)$ of type II there corresponds one of the following two 2-skeletons of type II:



where n and m_1 in $\mathfrak{S}_1$ are positive integers and n, m_1 and m_2 in $\mathfrak{S}_2$ are positive integers such that $\max(n, 2m_2) \leq 2m_1$.

The explicit form of $D(C(n+2), F)$ which corresponds to $\mathfrak{S}_1$ is determined in [4],[8]. We will here consider the case of $\mathfrak{S}_2$. We denote by $O(n)$ (resp. $U(n)$) the real orthogonal (resp. unitary) group of degree n and by E_n the unit matrix of degree n. Let $\{T_1, \cdots, T_n\}$ be a system of $m_1 \times m_2$-complex matrices T_k $(1 \leq k \leq n)$ satisfying the following condition:

(2) $\quad {}^t\overline{T}_k T_k = E_{m_2}$ $(1 \leq k \leq n)$, $\quad {}^t\overline{T}_k T_l + {}^t\overline{T}_l T_k = 0$ $\quad (1 \leq k \neq l \leq n)$.

Suppose that $\{T'_1, \cdots, T'_n\}$ is another system satisfying (2). Then $\{T_1, \cdots, T_n\}$ is said to be *equivalent* to $\{T'_1, \cdots, T'_n\}$ if there exists a triple $\{O_1, U_1, U_2\} \in O(n) \times U(m_1) \times U(m_2)$ such that

$$(T_1, \cdots, T_n) = U_1(T'_1, \cdots, T'_n)(O_1 \otimes U_2)$$

for the $m_1 \times nm_2$-matrices $(T_1, \cdots, T_n)$ and $(T'_1, \cdots, T'_n)$.

Let $\{T_1, \cdots, T_n\}$ be a system satisfying (2) and $W = \boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2}$ be the direct sum of the complex number spaces $\boldsymbol{C}^{m_i}$ $(i = 1, 2)$. Then we can define a $C(n+2)$-hermitian form F on W as follows;

$$F(u, u) = (\langle u_1, u_1 \rangle_1, \langle u_2, u_2 \rangle_2, \operatorname{Re}\langle u_1, T_1 u_2 \rangle_1, \cdots, \operatorname{Re}\langle u_1, T_n u_2 \rangle_1),$$

where $u = u_1 + u_2 \in W$ and $\langle\ ,\ \rangle_i$ is a canonical hermitian inner product in $\boldsymbol{C}^{m_i}$ $(i = 1, 2)$. Using the results on classification of N-algebras of type II [4] and Theorem A in [9], we have

Theorem 2. *For F above, the domain $D(C(n+2), F)$ is a homogeneous Siegel domain which corresponds to $\mathfrak{S}_2$. Conversely every homogeneous Siegel domain which corresponds to $\mathfrak{S}_2$ is constructed by the above way by taking some system $\{T_1, \cdots, T_n\}$ satisfying* (2). *Suppose that $D(C(n+2), F)$ (resp. $D(C(n+2), F')$) is constructed by $\{T_1, \cdots, T_n\}$ (resp. $\{T'_1, \cdots, T'_n\}$). Then $D(C(n+2), F)$ is holomorphically isomorphic to $D(C(n+2), F')$ if and only if $\{T_1, \cdots, T_n\}$ is equivalent to $\{T'_1, \cdots, T'_n\}$.*

Remark. If $m_1=m_2$ in $\mathfrak{S}_2$, then the condition (2) coincides with that of Pjateckii-Sapiro and the above construction of $D(C(n+2), F)$ is reduced to Pjateckii-Sapiro's [8].

3. As an application of Theorem 1 and Theorem 2, we get the following theorem. To prove this theorem we need mainly the results in [2], [5], [7], [8] and the well-known theorem of Borel-Koszul [1], [6].

Theorem 3. *The bounded symmetric domain in* $\boldsymbol{C}^{16}$ *of type* (V) (*in the sense of* E. Cartan) *is realized as* $D(C(8), F)$, *where* $F=(F_1, \cdots, F_8)$ *is the following* $C(8)$-*hermitian form on* $\boldsymbol{C}^8$:

$$F_1(u,u)=\sum_{1\leq k\leq 4}|u_k|^2, \qquad F_2(u,u)=\sum_{1\leq k\leq 4}|u_{k+4}|^2,$$
$$F_3(u,u)=\mathrm{Re}\,(u_1\bar{u}_5+u_2\bar{u}_6+u_3\bar{u}_7+u_4\bar{u}_8),$$
$$F_4(u,u)=\mathrm{Im}\,(-u_1\bar{u}_5+u_2\bar{u}_6+u_3\bar{u}_7-u_4\bar{u}_8),$$
$$F_5(u,u)=\mathrm{Re}\,(-u_1\bar{u}_6+u_2\bar{u}_5-u_3\bar{u}_8+u_4\bar{u}_7),$$
$$F_6(u,u)=\mathrm{Im}\,(u_1\bar{u}_6+u_2\bar{u}_5+u_3\bar{u}_8+u_4\bar{u}_7),$$
$$F_7(u,u)=\mathrm{Re}\,(-u_1\bar{u}_7+u_2\bar{u}_8+u_3\bar{u}_5-u_4\bar{u}_6),$$
$$F_8(u,u)=\mathrm{Im}\,(u_1\bar{u}_7-u_2\bar{u}_8+u_3\bar{u}_5-u_4\bar{u}_6),$$

where $u=(u_1, \cdots, u_8)\in \boldsymbol{C}^8$.

Remark. It has already been stated in [8] without proof that the Siegel domain isomorphic to the bounded symmetric domain in $\boldsymbol{C}^{16}$ of type (V) may be obtained from the skeleton $\mathfrak{S}_2$ with $(n, m_1, m_2)=(6, 4, 4)$.

As a corollary to Theorems 1, 2, 3, we have

Proposition. *Let* $D(C(n+2), F)$ *be a homogeneous Siegel domain which corresponds to* $\mathfrak{S}_2$ *with* $m_1=m_2$, $n\neq 2$, $(n, m_1)\neq(4,2)$ *and* $(n, m_1)\neq(6,4)$. *Then* $\mathfrak{g}_h=\mathfrak{g}_a$.

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