# 115. *Characterizations of Compactness and Countable Compactness*

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It is known that if a topological space $Y$ is compact, then the following condition is satisfied.

(*) For every topological space $X$, each mapping of $X$ into $Y$ with closed graph is continuous.

The purpose of this note is to show that this condition characterizes compact spaces among $T_1$ spaces by proving somewhat strengthened result. A similar characterization of countably compact spaces is also stated.

Recall that a net in a set $X$ is an ordered pair $(f, (D, \leqslant))$ of a directed set $(D, \leqslant)$ and a mapping $f$ of $D$ into $X$. If $a$ is an element of a directed set $(D, \leqslant)$, we denote by $D(a)$ the set of all $x \in D$ with $a \leqslant x$.

Let $\mathcal{S}$ be a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces. Thus for example $\mathcal{S}$ may be the class of Hausdorff completely regular spaces or that of paracompact spaces. We have the following

**Theorem 1.** *A $T_1$ topological space $Y$ is compact if and only if for every topological space $X$ belonging to $\mathcal{S}$, each mapping of $X$ into $Y$ with closed graph is continuous.*

**Proof.** Only the proof of the "if" part is needed. Suppose that $Y$ is not compact. Then there is a net $(f, (D, \leqslant))$ in $Y$ which has no adherent point. Let $\infty \notin D$, and let $X = D \cup \{\infty\}$. It is easy to see that the family $\mathcal{P}(D) \cup \{D(x) \cup \{\infty\} \mid x \in D\}$ is a base for a topology $\tau$ on $X$, where $\mathcal{P}(D)$ denotes the power set of $D$.

To prove that $\tau$ is Hausdorff, it suffices to show that for every $x \in D$, there is an element $y \in D \backslash \{x\}$ with $x \leqslant y$, since this implies $\{x\} \cap (D(y) \cup \{\infty\}) = \emptyset$. To this end suppose the contrary: there is an $x \in D$ such that $x \leqslant y$ does not hold for any $y \in D \backslash \{x\}$. If $y \in D$, then we have $x \leqslant z$ and $y \leqslant z$ for some $z \in D$, and consequently $z = x$ and $y \leqslant x$. Therefore we have $y \leqslant x$ for all $y \in D$, which yields however a contradiction that $f(x)$ is an adherent point of the net $(f, (D, \leqslant))$.

Let us proceed to prove that $(X, \tau)$ is completely normal. Let $A$ and $B$ be separated subsets of $X$, i.e., $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. If $\infty \notin \overline{A}$, then $\overline{A}$ and $\overline{A}^c = X \backslash \overline{A}$ are open disjoint and $B \subset \overline{A}^c$. If $\infty \notin B$, then $B$

and $\bar{B}^c$ are open disjoint and $A \subset \bar{B}^c$. Thus $(X, \tau)$ is completely normal.

Moreover $(X, \tau)$ is fully normal. In fact, each open cover $\mathcal{O}$ of $X$ contains a member $G$ with $\infty \in G$, and hence the open cover $\{\{x\} \mid x \in D\} \cup \{D(a) \cup \{\infty\}\}$ of $X$, where $a$ is an element of $D$ such that $D(a) \cup \{\infty\} \subset G$, is a star refinement of $\mathcal{O}$, as can be readily verified.

Now let $b$ be an element of $Y$, and consider the mapping $f^*$ of $X$ into $Y$ defined by $f^*(\infty) = b$ and $f^*(x) = f(x)$ for every $x \in D$. To complete the proof, it is enough to show that the graph $G(f^*)$ of $f^*$ is closed but $f^*$ is not continuous. Let $(x, y) \in (X \times Y) \setminus G(f^*)$. The set $U = Y \setminus \{f^*(x)\}$ is a neighborhood of $y$. Hence if $x \in D$, then $\{x\} \times U$ is a neighborhood of $(x, y)$ which is disjoint from $G(f^*)$. If $x = \infty$, then since $y$ is not an adherent point of the net $(f, (D, \leqslant))$, we can find a neighborhood $V \subset X \setminus \{b\}$ of $y$ and an $a \in D$ such that $V \cap f(D(a)) = \emptyset$, and so we have $((D(a) \cup \{\infty\}) \times V) \cap G(f^*) = \emptyset$, which shows that $G(f^*)$ is closed. On the other hand, the identity mapping $e$ of $D$ into itself constitutes, together with $(D, \leqslant)$, a net $(e, (D, \leqslant))$ in $X$ which obviously converges to $\infty$. However $f = f^* \circ e$ can not converge to $b = f^*(\infty)$. This completes the proof.

As can easily be seen, a similar argument establishes the implication (3)$\Rightarrow$(1) of the following theorem, in which $\bar{N}$ denotes the one-point compactification of $N$, the set of all positive integers, with the discrete topology. The implication (1)$\Rightarrow$(2) is due to P. E. Long.[*]

**Theorem 2.** *For a $T_1$ topological space $Y$, the following conditions are equivalent.*

(1) *$Y$ is countably compact.*

(2) *For every first countable topological space $X$, each mapping of $X$ into $Y$ with closed graph is continuous.*

(3) *Each mapping of $\bar{N}$ into $Y$ with closed graph is continuous.*

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*) P. E. Long: Functions with closed graphs. Amer. Math. Monthly, **76**, 930–932 (1969).