106. Note on Potential Operators on L^p

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The purpose of the present paper is to prove in an abstract setting a theorem on the existence and non-existence in L^p $(1\leqslant p<\infty)$ of potential operators associated with a (temporally homogeneous) Markov process with an invariant measure. We shall apply this result to a consideration of abstract "semi-linear Poisson's equations" (cf. Konishi [10]) in L^1 and L^2.

Remember that an equi-continuous semi-group $\{T_t\}_{t\geqslant 0}$ of class (C_0) in a Banach space X is said to admit a *potential operator* V (in the sense of Yosida [19]) if its infinitesimal generator A admits a densely defined inverse A^{-1}: $V=-A^{-1}$ (see also Yosida [21] and Chapter XIII, 9 of Yosida [22]). We shall make use of the fact that $\{T_t\}_{t\geqslant 0}$ admits a potential operator if and only if $\lim_{\lambda\downarrow 0}\lambda(\lambda I-A)^{-1}f=0$ for every $f\in X$. (See also Theorem 2.2 of Sato [14] for several other criteria for the existence of potential operators.)

1. Potential operators on L^p. Let $\mathfrak{B}$ be a σ-additive family of subsets of a set $S\neq\phi$ and $P(t,x,E)$, $t>0$, $x\in S$, $E\in\mathfrak{B}$, be the transition probability of a Markov process on the phase space $(S,\mathfrak{B})$ with a (σ-finite) invariant positive measure m (see, e.g., Yosida [22], XIII, 1). Then by the relation:

$$(T_{p,t}f)(x)=\int_S P(t,x,dy)f(y),\qquad f\in L^p\equiv L^p(S,\mathfrak{B},m),$$

a non-negative contraction semi-group $\{T_{p,t}\}_{t>0}$ in real L^p is defined for each $1\leqslant p\leqslant\infty$. Let $\mathscr{B}$ be a closed subspace of L^∞ such that

(1) $\qquad \mathscr{B}\cap L^1$ is dense in L^p, $\qquad 1<p<\infty$,

and that $T_{\infty,t}$, $t>0$, leaves $\mathscr{B}$ invariant. Denote by $T_{\mathscr{B},t}$, $t>0$, the restriction of $T_{\infty,t}$ to $\mathscr{B}$. We assume that the semi-group

(2) $\qquad \{T_{\mathscr{B},t}\}_{t\geqslant 0}\subset L(\mathscr{B},\mathscr{B})$ is of class (C_0)

and, moreover, that the semi-groups

(3) $\qquad \{T_{p,t}\}_{t\geqslant 0}\subset L(L^p,L^p)$, $1\leqslant p<\infty$, are of class (C_0).

We denote by $A_{\mathscr{B}}$ and A_p's their infinitesimal generators respectively.

Theorem. *Suppose that the semi-group*

(4) $\qquad \{T_{\mathscr{B},t}\}_{t\geqslant 0}$ *admits a potential operator.*

Then we have the following:

(i) *The semi-group* $\{T_{p,t}\}_{t\geqslant 0}$ *in* L^p, $1<p<\infty$, *admits a potential operator.*

(ii) *The operator A_1 in L^1 is one-to-one but its range is not dense in L^1.*

Proof. *Proof of* (i). Set $f \in \mathscr{B} \cap L^1$. Then for $1<p<\infty$,

$$\begin{aligned} &\|\lambda(\lambda I-A_p)^{-1}f\|_p \qquad (\|\cdot\|_p=\|\cdot\|_{L^p}) \\ &\quad\leqslant \|\lambda(\lambda I-A_{\mathscr{B}})^{-1}f\|_\infty^{(p-1)/p}\cdot\|\lambda(\lambda I-A_1)^{-1}f\|_1^{1/p} \\ &\quad\leqslant \|\lambda(\lambda I-A_{\mathscr{B}})^{-1}f\|_\infty^{(p-1)/p}\cdot\|f\|_1^{1/p}. \end{aligned}$$

Thus

$$\lim_{\lambda\downarrow 0}\lambda(\lambda I-A_p)^{-1}f=0 \qquad \text{in } L^p.$$

In view of (1), this holds for every $f \in L^p$.

Proof of (ii). Suppose that $A_1u=0$ for some $u \in D(A_1)$. Then $(\lambda I-A_1)u=\lambda u$ whenever $\lambda>0$. Thus

$$\begin{aligned} |u|&=|\lambda(\lambda I-A_1)^{-1}u|\leqslant\lambda(\lambda I-A_1)^{-1}|u| \\ &=\lambda(\lambda I-A_1)^{-1}(|u|-n)^+ +\lambda(\lambda I-A_2)^{-1}(|u|\wedge n) \end{aligned}$$

for every $n=1,2,\cdots$. Therefore

$$\begin{aligned} &\int_E |u(x)|\,m(dx), \qquad E\in\mathfrak{B},\ m(E)<+\infty, \\ &\quad\leqslant\|\lambda(\lambda I-A_1)^{-1}(|u|-n)^+\|_1+\int_E(\lambda(\lambda I-A_2)^{-1}(|u|\wedge n))(x)m\,(dx) \\ &\quad\leqslant\|(|u|-n)^+\|_1+\sqrt{m(E)}\,\|\lambda(\lambda I-A_2)^{-1}(|u|\wedge n)\|_2 \end{aligned}$$

Since $\lambda>0$ is arbitrary, we obtain, in view of Theorem, (i), that

$$\int_E|u(x)|\,m(dx)\leqslant\|(|u|-n)^+\|_1,$$

which implies $\|u\|_1\leqslant\|(|u|-n)^+\|_1$, $n=1,2,\cdots$. Thus $\|u\|_1=0$. That is, $u=0$. Next we shall show that $R(A_1)$ is not dense in L^1, i.e., $\{T_{1,t}\}_{t\geqslant 0}$ does not admit a potential operator. Since $L^1\neq\{0\}$, there exists $f_0\in L^1$ satisfying $f_0\geqslant 0$, $f_0\neq 0$, for which we have

$$\begin{aligned} \|\lambda(\lambda I-A_1)^{-1}f_0\|_1&=\left\|\lambda\int_0^\infty e^{-\lambda t}T_{1,t}f_0\,dt\right\|_1 \\ &=\lambda\int_0^\infty e^{-\lambda t}\|T_{1,t}f_0\|_1dt=\|f_0\|_1>0. \end{aligned}$$

Q.E.D.

Remark 1. Consider, in particular, a Feller process on a locally compact (non-compact) Hausdorff space with a countable base and set $\mathscr{B}=C_0(S)$: the space of real-valued continuous functions on S vanishing at infinity. Thus the condition (2) is satisfied.

(a) The assumption (4) of our theorem is fulfilled in many examples since Sato [14] states that the semi-group $\{T_{\mathscr{B},t}\}_{t\geqslant 0}$ admits a potential operator if the process is "transient" or "null recurrent" (Theorems 3.1 and 3.2);

(b) According to Theorem 3.3 of Sato [14], the assumption of our theorem implies that $m(S)=+\infty$.

(See also a recent note of Revuz [12].)

Example. We can apply our theorem to a Lévy process on $S=R^N$ $(N\geqslant 1)$ with $\mathscr{B}=C_0(R^N)$ and $m=dx$. In this case (1) and (2) are true.

In view of Theorem 3.2 of Watanabe [16], (3) is fulfilled. Moreover, if $A_{\mathcal{B}} \neq 0$, (4) is satisfied according to Sato [14] (Theorem 4.1) or Hirsch [8] (Theorem 4) (see also Yosida [20] for the case of the N-dimensional Brownian motion). Consequently we have an operator-theoretical proof of a part of Theorem 1.5 of Watanabe [17].

2. Abstract semi-linear Poisson's equations in L^1. Let β be a maximal monotone graph in $R^1 \times R^1$ which contains the origin. Define the corresponding m-accretive (multi-valued) operator β_1 in L^1 by:

$$f \in \beta_1 u \quad \text{iff} \quad (u(x), f(x)) \in \beta \quad m\text{-a.e. } x \in S. \tag{5}$$

Then the operator $-A_1 + \beta_1$ is m-accretive in L^1 by Theorem 1 of Brezis and Strauss [4]. Moreover we obtain the following.

Corollary 1. *Under the assumption of Theorem, the (possibly) multi-valued operator* $-A_1 + \beta_1$ *admits a single-valued inverse* $(-A_1 + \beta_1)^{-1}$ *in* L^1.

Proof. Set $h \in R(-A_1 + \beta_1)$. Suppose that

$$-A_1 u + \beta_1 u \ni h \quad \text{and} \quad -A_1 \hat{u} + \beta_1 \hat{u} \ni h$$

for some pair $u, \hat{u} \in D(A_1) \cap D(\beta_1)$. Thus, there exist $w \in \beta_1 u$ and $\hat{w} \in \beta_1 \hat{u}$ such that

$$-A_1 u + w = h \quad \text{and} \quad -A_1 \hat{u} + \hat{w} = h.$$

Since the operator A_1 is dissipative (s) (see Sato [13]), we have

$$\tau(u - \hat{u}, w - \hat{w}) = \tau(u - \hat{u}, A_1 u - A_1 \hat{u}) \leqslant 0;$$

where, by definition, $\tau(f, g) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (\|f + \varepsilon g\|_1 - \|f\|_1)$. In view of the concrete form of τ due to Sato [13]:

$$\tau(f, g) = \int_{\{x;\, f(x) \neq 0\}} (\operatorname{sgn} f(x)) g(x) m(dx) + \int_{\{x;\, f(x) = 0\}} |g(x)|\, m(dx),$$

we easily obtain that $w = \hat{w}$. Therefore $A_1 u = A_1 \hat{u}$. Accordingly, by means of (ii) of Theorem, $u = \hat{u}$. Q.E.D.

Remark 2. It has been pointed out by Crandall [5] and Konishi [9] (cf. also Konishi [11] and Bénilan [2]), that the study of the concentration-dependent diffusion equation (see, e.g., Ames [1], 1.2):

$$(*) \qquad \begin{aligned} \partial u / \partial t &= \operatorname{div}(D(u) \operatorname{grad} u), \qquad D(r) \geqslant 0 \ (r \in R^1), \\ &= \Delta \varphi(u), \qquad \varphi(u) = \int_0^u D(r) dr, \end{aligned}$$

leads to that of

$$u - \lambda \Delta \varphi(u) = f \in L^1, \lambda > 0,$$

which, via the "Kirchhoff transformation" (see Ames [1], p. 21): $\varphi(u) = v$, is equivalent to

$$\varphi^{-1}(v) - \lambda \Delta v \ni f.$$

Thus the author believes that further detailed investigations in semi-linear Poisson's equations, say, in $L^1(R^N)$ will give us informations for studies of (*) within the framework of the Crandall-Liggett theory ([6]).[1]

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1) The author thanks Prof. M. G. Grandall for his personal brief communication on $u_t = \Delta \varphi(u)$ in R^N.

3. Abstract semi-linear Poisson's equations in L^2. Let β be as in § 2 and define the corresponding operator β_2 in L^2 as in (5). It is known that the operator $-A_2+\beta_2$ is m-accretive (maximal monotone).

Corollary 2. *Suppose either one of the following:*

(i) *β is single-valued;*

(ii) *A_2 is self-adjoint.*

Then under the assumption of Theorem, $-A_2+\beta_2$ admits a single-valued densely-defined inverse $(-A_2+\beta_2)^{-1}$.

Proof. In either case the denseness of $D((-A_2+\beta_2)^{-1})$ is established as in Konishi [10]. In the case (i), the single-valuedness of $(-A_2+\beta_2)^{-1}$ may be proved as in the proof of Corollary 1. Now we shall show the single-valuedness of $(-A_2+\beta_2)^{-1}$ in the case (ii). Set $h \in R(-A_2+\beta_2)$. Suppose that

$$-A_2u+\beta_2u \ni h \quad \text{and} \quad -A_2\hat{u}+\beta_2\hat{u} \ni h$$

for some pair $u, \hat{u} \in D(A_2) \cap D(\beta_2)$. Then

$$\begin{aligned}\|\sqrt{-A_2}(u-\hat{u})\|_2^2 &= -(A_2(u-\hat{u}), u-\hat{u}) \\ &= -((A_2u+h)-(A_2\hat{u}+h), u-\hat{u}) \leqslant 0.\end{aligned}$$

Thus $\sqrt{-A_2}(u-\hat{u})=0$, which yields $A_2(u-\hat{u})=0$. Accordingly $u=\hat{u}$.

Q.E.D.

Remark 3.[2)] Since the operator $-A_2+\beta_2$ is maximal monotone, so is $V_\beta \equiv (-A_2+\beta_2)^{-1}$. Noticing the relation:

$$(\lambda I-A_2+\beta_2)^{-1}=\frac{1}{\lambda}\{I-(I+\lambda V_\beta)^{-1}\}, \qquad \lambda>0,$$

(which seems to be a nonlinear version of, say, (3) of Sato [15]) and using Theorem 2.3 of Crandall and Pazy [7] or Proposition 2.6 of Brézis [3], we obtain the following, which relies entirely upon the structure of the Hilbert space L^2.

Under the assumption of Corollary 2, we can conclude the following:

(a) *For all $f \in D(V_\beta)$*

$$\|(\lambda I-A_2+\beta_2)^{-1}f\|_2 \uparrow \|V_\beta f\|_2$$

when $\lambda \downarrow 0$ and

$$\lim_{\lambda \downarrow 0} (\lambda I-A_2+\beta_2)^{-1}f=V_\beta f \tag{6}$$

with

$$\|(\lambda I-A_2+\beta_2)^{-1}f-V_\beta f\|_2^2 \leqslant \|V_\beta f\|_2^2-\|(\lambda I-A_2+\beta_2)^{-1}f\|_2^2$$

(b) *For $f \notin D(V_\beta)$,*

$$\|(\lambda I-A_2+\beta_2)^{-1}f\|_2 \uparrow +\infty$$

when $\lambda \downarrow 0$.

Thus in particular

$$D(V_\beta)=\{f \in L^2 ; \lim_{\lambda \downarrow 0} (\lambda I-A_2+\beta_2)^{-1}f \text{ exists}\}. \tag{6$'$}$$

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2) The author thanks Prof. M. G. Crandall and Prof. A. Pazy for their kind advices on the original version of this remark.

The formula (6) may be considered to be a nonlinear version of another definition of a potential operator (see (7) of Yosida [22], p. 412, or Yosida [18]).

References

[1] W. F. Ames: Nonlinear Partial Differential Equations in Engineering. Academic Press, New York - London (1965).

[2] Ph. Bénilan: Equations d'évolution dans un espace de Banach quelconque et applications. Thesis, Orsay (1972).

[3] H. Brézis: Opérateurs maximaux monotones et semi-groupes de contractions dans le espaces de Hilbert. North-Holland/American Elsevier (1973).

[4] H. Brezis and W. A. Strauss: Semi-linear second-order elliptic equations in L^1. J. Math. Soc. Japan (to appear).

[5] M. G. Crandall: Semigroups of nonlinear transformations in Banach spaces. Contributions to Nonlinear Functional Analysis, 157–179 (1971).

[6] M. G. Crandall and T. M. Liggett: Generation of semi-groups of nonlinear transformations on general Banach spaces. Amer. J. Math., **93**, 265–298 (1971).

[7] M. G. Crandall and A. Pazy: Semi-groups of nonlinear contractions and dissipative sets. J. Functional Analysis, **3**, 376–418 (1969).

[8] F. Hirsch: Opérateurs codissipatifs et dissipatifs invariants par translations. C. R. Acad. Sc. Paris, Série A, **272**, 231–233 (1971).

[9] Y. Konishi: Some examples of nonlinear semi-groups in Banach lattices. J. Fac. Sci. Univ. Tokyo, Sect. IA, **18**, 537–543 (1972).

[10] ——: Semi-linear Poisson's equations. Proc. Japan Acad., **49**, 100–105 (1973).

[11] ——: On the nonlinear semi-groups associated with $u_t = \Delta\beta(u)$ and $\varphi(u_t) = \Delta u$. J. Math. Soc. Japan (to appear).

[12] D. Revuz: Sur les opérateurs potentiels au sens de Yosida. Z. Wahrscheinlichkeitstheorie verw. Geb., **25**, 199–203 (1973).

[13] K. Sato: On the generators of non-negative contraction semi-groups in Banach lattices. J. Math. Soc. Japan, **20**, 423–436 (1968).

[14] ——: Potential operators for Markov processes. Proceedings of the Sixth Kerkeley Symposium on Mathematical Statistics and Probability, **3**, 193–211 (1972).

[15] ——: A note on infinitesimal generators and potential operators of contraction semigroups. Proc. Japan Acad., **48**, 450–453 (1972).

[16] T. Watanabe: Some potential theory of processes with stationary independent increments by means of the Schwartz distribution theory. J. Math. Soc. Japan, **24**, 213–231 (1972).

[17] ——: Some recent results on processes with stationary independent increments. Second Japan-USSR Symposium on Probability Theory, **2**, 93–110 (1972).

[18] K. Yosida: Positive resolvents and potentials. Z. Wahrscheinlichkeitstheorie verw. Geb., **8**, 210–218 (1967).

[19] ——: The existence of the potential operator associated with an equicontinuous semigroup of class (C_0). Studia Math., **31**, 531–533 (1968).

[20] ——: On the potential operators associated with Brownian motions. J. Analyse Math., **23**, 461–465 (1970).

[21] K. Yosida: Abelian ergodic theorems and Hunt's theory of potentials (in Japanese). Sûgaku, **22**, 81–91 (1970).
[22] ——: Functional Analysis. Springer, Berlin - Heidelberg - New York (1971).