104. *A Typical Formal Group in K-Theory*

By Shôrô ARAKI
Osaka City University

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Typical formal groups were defined by Cartier [4] and used by Quillen [9] to decompose U-cobordism, localized at a prime p, into a direct sum of Brown-Peterson cohomologies with shifted degrees.

On the other hand, complex K-theory, localized at a prime p, was decomposed into $p-1$ factors by Adams [1] and Sullivan [11]. This decomposition is given in [1] with explicit idempotents. Its central factor inherits a multiplicative structure from K-theory so that we can expect a related formal group. In the present note the author observes that the desired formal group is in fact a typical group law with a simple nature.

As an application, using this typical formal group and a description of the polynomial basis of $BP^*(pt)$ (Theorem 1), we obtain a proof of Stong-Hattori theorem based on formal group techniques.

The details will appear elsewhere.

1. Typical formal groups. Let R be a commutative ring with unity and F a (one-dimensional) commutative formal group over R. A formal power series γ over R without constant term is called a *curve* over F. The addition $\gamma +_F \gamma'$ of two curves over F is defined by

$$(\gamma +_F \gamma')(T) = F(\gamma(T), \gamma'(T)).$$

With this addition the set C_F of all curves over F forms an abelian group. On C_F 3 kinds of operators are defined [4] by the following formulas:

i) $(\boldsymbol{f}_n\gamma)(T) = \sum_{k=1}^{n}{}_F\, \gamma(\zeta_k T^{1/n}), \qquad n \geqq 1,$

where $\zeta_k = \exp 2\pi k\sqrt{-1}/n$, the n-th roots of unity;

ii) $(\boldsymbol{v}_n\gamma)(T) = \gamma(T^n), \qquad n \geqq 1;$

iii) $([a]\gamma)(T) = \gamma(aT), \qquad a \in R.$

Operators $\boldsymbol{f}_n$ are called *Frobenius operators* and particularly important. These 3 kinds of operators satisfy certain universal relations [4], and we treat C_F as an operator-module. A curve γ_0 defined by $\gamma_0(T) = T$ will be regarded as the one of the basic curves.

Some functorialities of these operator-modules should be observed. Let F and G be formal groups over R and $\varphi : F \to G$ a homomorphism, i.e., a curve over G satisfying

$$\varphi \circ F = G \circ (\varphi \times \varphi).$$

Then

$$\varphi_{\#}: C_F \to C_G$$

defined by $\varphi_{\#}\gamma = \varphi \circ \gamma$ is a homomorphism of operator-modules. Next let $\theta: R \to S$ be a homomorphism of commutative rings with unity and F a formal group over R. $\theta_* F$ is a formal group over S induced from F by coefficient homomorphism θ. Then

$$\theta_*: C_F \to C_{\theta_* F}$$

obtained by coefficient homomorphism θ is also a homomorphism of operator-modules.

Let p be a fixed prime. A curve γ over F is called *typical* when $\boldsymbol{f}_q \gamma = 0$ for all $q > 1$ such that $(q, p) = 1$. The formal group F is called *typical* when γ_0 is typical [4]. Typical curves and formal groups are mostly observed when the ground ring R is a $Z_{(p)}$-algebra, where $Z_{(p)}$ denotes integers localized at the prime p. In this case Cartier defined an idempotent

$$\varepsilon_F: C_F \to C_F$$

by

$$\varepsilon_F = {\sum_{(n,p)=1}}_F \left(\frac{\mu(n)}{n}\right)_F \boldsymbol{v}_n \boldsymbol{f}_n$$

where μ is the Möbius function. A curve γ is typical iff $\gamma \in \operatorname{Im} \varepsilon_F$. In particular

(1) $$\xi_F = \varepsilon_F \gamma_0$$

is a typical curve over F which we regard as the *canonical* typical curve over F.

Let $\gamma \in C_F$ be *invertible* with respect to composition. As usual we define another formal group F^γ by

$$F^\gamma = \gamma^{-1} \circ F \circ (\gamma \times \gamma).$$

Then $\gamma: F^\gamma \rightrightarrows F$, a (weak) isomorphism, and it is a strict isomorphism when $\gamma(T) = T + \text{higher terms}$. We remark that F^γ is typical iff γ is typical. Thus, when R is a $Z_{(p)}$-algebra, we have a standard way to associate with each formal group F over R a typical formal group F^{ξ_F} which is strictly isomorphic to F. We regard F^{ξ_F} as the typical group law *canonically associated* to F.

In fact, Quillen [9] used this construction of typical formal group in case $F = F_U$, the formal group of U-cobordism, and we use the same construction in case $F = F_K$, the formal group of K-theory.

We need a remark about typical curves over typical group laws. Let R be a $Z_{(p)}$-algebra and μ a typical formal group over R. Every typical curve over μ can be expressed uniquely as a Cauchy series

(2) $$\gamma(T) = {\sum_{k \geq 0}}_\mu a_k T^{p^k}, \qquad a_k \in R.$$

2. A polynomial basis of $BP^*(pt)$. Let R be a $Z_{(p)}$-algebra, μ a typical group law over R, and assume that p is not a zero-divisor of R.

$f_p\gamma_0$ is a typical curve over μ and we see easily that $f_p\gamma_0=0$ iff μ is additive. Thus $f_p\gamma_0$ is a measure of deviation of μ from additive group law or an obstruction to identify μ with an additive one. And, expressing as

$$(f_p\gamma_0)(T)=\sum_{k\geqq 0} {}_{\mu}v_{k+1}T^{p^k}$$

by (2), we obtain a series of obstruction elements $v_1, v_2, \cdots$.

Now we consider the case of $\mu=\mu_{BP}$, the formal group of Brown-Peterson cohomology. We remark that this is a typical group law [9] and universal for typical group laws over $Z_{(p)}$-algebras. Thus, in this case $\{v_1, v_2, \cdots\}$ are universal obstructions to additivity.

Theorem 1. *Let $f_{n,BP}$ denote the Frobenius operators of μ_{BP} and put*

$$(f_{p,BP}\gamma_0)(T)=\sum_{k\geqq 0} {}_{\mu_{BP}}v_{k+1}T^{p^k}.$$

Then the coefficients $\{v_1, v_2, \cdots\}$ form a polynomial basis of the polynomial algebra $BP^*(pt)$ with $\deg v_i=-2(p^i-1)$, $i\geqq 1$.

Let $\log_{BP}$ be the logarithm of μ_{BP}, i.e., $\log_{BP}: \mu_{BP}\cong G_a$ (additive group law), the strict isomorphism over the rationals Q. Compute $\log_{BP\#}f_{p,BP}\gamma_0$ in two ways and compare the coefficient of each power T^{p^k}, then we obtain a recursive formula which describe the relations between the above generators v_i and the coefficients of $\log_{BP}$. Obtained formula is the same as the formula given by Hazewinkel [7]. Thus our polynomial basis of $BP^*(pt)$ is the same as those given by Hazewinkel. Cf., also Liulevicius [8] for the case $p=2$.

3. Formal groups of K-theory. We shall discuss the formal groups of complex K-theory. For complex K-functor we use

$$\lambda_{-1}(E)=\sum_i (-1)^i\lambda^i(E)=e^K(E)$$

as the Euler class of the vector bundle E. Thus, for a line bundle L we have

$$e^K(L)=1-L$$

so that the corresponding formal group is

$$F_K(X, Y)=X+Y-XY=1-(1-X)(1-Y).$$

On this formal group we remark two facts: the Frobenius operators satisfy

$$f_{n,K}\gamma_0=\gamma_0$$

for all $n\geqq 1$; and over Q the logarithm $\log_K: F_K\cong G_a$ is described by

$$\log_K T=-\log(1-T)=\sum_{n\geqq 1}\frac{T^n}{n}.$$

Now localize at a prime p. Over $Z_{(p)}$ the canonical typical curve is given by

$$\xi_K(T)=(\varepsilon_K\gamma_0)(T)=1-P(1-T)$$

where $P(1-T)=\prod_{(m,p)=1}(1-T^m)^{\mu(m)/m}$ is the power series of Hasse [5].

Put

$$L(1-T)=\sum_{k\geq 0}\frac{1}{p^k}T^{p^k}$$

and remark the following relation [5]

$$L(1-T)=-\log P(1-T).$$

Let $\mu_K=F_K^{\xi_K}$, the typical group law canonically associated to F_K. Then

$$\log_{\mu_K}=\log_K\circ\xi_K$$

over Q. Hence we have

$$\log_{\mu_K}(T)=L(1-T)=\sum_{k\geq 0}\frac{1}{p^k}T^{p^k}.$$

Next we observe formal groups of periodic K-cohomology $K^*(X)$. Its coefficient object is $K^*(pt)=Z[u,u^{-1}]$, where $u\in K^{-2}(pt)$ is the Bott periodicity element. For our purpose it is convenient to choose the K^*-theoretic Euler class of a line bundle L so as to lie in $K^2(X)$, i.e.,

$$e^{K^*}(L)=u^{-1}\cdot e^K(L).$$

The corresponding formal group is

$$F_{K^*}(X,Y)=X+Y-u\cdot XY$$

with the logarithm

$$\log_{K^*}(T)=-u^{-1}\log(1-uT)=\sum_{n\geq 1}\frac{1}{n}u^{n-1}T^n.$$

After localized at the prime p, the canonical typical curve $\xi_{K^*}=\varepsilon_{K^*}\gamma_0$ is given by

$$\xi_{K^*}(T)=u^{-1}P(1-uT)$$

over $K^*(pt)_{(p)}$. Let $\mu_{K^*}=F_{K^*}^{\xi_{K^*}}$, the canonically associated typical formal group. Its logarithm is given by

$$\log_{\mu_{K^*}}(T)=u^{-1}L(1-uT)=\sum_{k\geq 0}\frac{1}{p^k}u^{p^k-1}T^{p^k}.$$

4. The formal group of $G^*(X)$. Fix a prime p. Adams [1] defined additive idempotents

$$E_s: K(X)_{(p)}\to K(X)_{(p)}$$

of K-theory localized at the prime p for $s\in Z$, which depends actually only on the coset "s mod $p-1$". E_s's decompose $K(X)_{(p)}$ into the natural direct sum

$$K(X)_{(p)}=E_0K(X)_{(p)}+\cdots+E_{p-2}K(X)_{(p)}.$$

As to the basic properties of these idempotents, cf., [1].

These idempotents give rise to an idempotent

$$E_K: K^*(X)_{(p)}\to K^*(X)_{(p)}$$

of the periodic K-cohomology by the requirements: (i) E_K is stable and (ii) the following diagram

$$\begin{array}{ccc} \tilde{K}^{2i}(X)_{(p)} & \xrightarrow{\beta^i} & \tilde{K}(X)_{(p)} \\ \downarrow E_K & & \downarrow E_i \\ \tilde{K}^{2i}(X)_{(p)} & \xrightarrow{\beta^i} & K(X)_{(p)} \end{array}$$

commutes for all $i \in Z$, where β is the Bott periodicity, i.e., the multiplication with u. We put

$$G^*(X) = E_K K^*(X)_{(p)}.$$

It turns out that i) $G^*(X)$ inherits its multiplicative structure from $K^*(X)$, ii) $G^*(pt) = Z_{(p)}[u_1, u_1^{-1}]$ such that $u_1 = u^{p-1}$, i.e., $G^*(X)$ is a periodic cohomology theory of period $2(p-1)$ with u_1 as the periodicity element.

Theorem 2. $td(e^{BP}(L)) = e^{\mu_{K^*}}(L) \in G^2(X)$
where $e^{BP}(L)$ and $e^{\mu_{K^*}}(L)$ denote Euler classes of a line bundle L corresponding to the formal groups μ_{BP} and μ_{K^*} respectively.

This theorem implies that

$$td(BP^*(X)) \subset G^*(X)$$

by a standard argument. Thus μ_{K^*} is already defined on $G^*(pt)$ and gives a typical formal group μ_{G^*} of G^*-theory corresponding to the Euler class $e^{G^*}(L) = e^{\mu_{K^*}}(L)$.

5. Stong-Hattori Theorem. Here we put $\mu = \mu_{G^*}$. Let $\boldsymbol{t} = (t_1, t_2, \cdots)$ be a sequence of indeterminates with deg $t_j = -2(p^j - 1)$. We put

$$\phi_t(T) = \sum_{j \geqq 0} {}_{\mu} t_j T^{p^j}, \qquad t_0 = 1.$$

ϕ_t is a typical curve of μ over $G^*(pt)[\boldsymbol{t}]$ and invertible. Hence

$$\mu' = \mu^{\phi t}$$

is a typical group law over $G^*(pt)[\boldsymbol{t}]$. By the universality of μ_{BP} we get a unique homomorphism of graded algebras

$$h : BP^*(pt) \to G^*(pt)[\boldsymbol{t}]$$

such that $h_* \mu_{BP} = \mu'$. In fact, this map can be extended to arbitrary complexes so that it gives a cohomology map. By a standard argument we can identify h with the Boardman map

$$\pi_*(BP) \to \pi_*(G \wedge BP).$$

Thus we can state Stong-Hattori theorem [6, 10] as

Theorem 3. *h is an injection to a direct summand.*

Cf., also [3]. For the proof it is sufficient to prove that "h mod p" is injective.

Put

(3) $$h_*(f_{p,BP}\gamma_0)(T) = (f_{p,\mu'}\gamma_0)(T) = \sum_{i \geqq 1} {}_{\mu'} \bar{v}_i T^{p^{i-1}},$$

i.e., $\bar{v}_i = h(v_i)$ for $j \geqq 1$. Then

$$\phi_t \circ (f_{p,\mu'}\gamma_0) = \phi_{t\#}(f_{p,\mu'}\gamma_0) = f_{p,\mu}\phi_t,$$

hence

$$(f_{p,\mu'}\gamma_0)(T) = \phi_t^{-1}\Big(\sum_{j \geqq 0} {}_{\mu} f_{p,\mu}(t_j T^{p^j})\Big).$$

We compute $f_{p,\mu}(t_j T^{p^j})$ as follows:

$$f_{p,\mu}(t_0T)=u_1T$$

and

$$f_{p,\mu}(t_jT^{p^j})\equiv u_1t_j^pT^{p^j} \qquad \mathrm{mod}\, p$$

for $j>0$. We put

$$I=(t_1, t_2, \cdots),$$

the augmentation ideal of $G^*(pt)[t]$. Then

$$\phi_t((f_{p,\mu'}\gamma_0)(T))\equiv u_1T \qquad \mathrm{mod}\,(p)+I^2.$$

Here we remark that ϕ_t^{-1} is a typical curve of μ', and put

$$\phi_t^{-1}(T)=\sum_{j\geq 0}{}_{\mu'}s_jT^{p^j}, \qquad s_0=1.$$

Then $s_j\in I$ for $j>0$ and we obtain

(4) $$(f_{p,\mu'}\gamma_0)(T)\equiv\sum_{j\geq 0}{}_{\mu'}u_1^{p^j}s_jT^{p^j} \qquad \mathrm{mod}\,(p)+I^2.$$

On the other hand, by easy arguments with respect to typical formal groups we obtain

$$s_j+t_j\equiv 0 \qquad \mathrm{mod}\, I^2$$

for $j>0$. Thus

$$G^*(pt)[t]=G^*(pt)[s_1, s_2, \cdots]$$
$$=G^*(pt)[u_1^ps_1, u_1^{p^2}s_2, \cdots, u_1^{p^j}s_j, \cdots]$$

since u_1 is invertible.

Finally (3) and (4) show that

$$G^*(pt)[t]\otimes Z_p=G^*(pt)[\bar{v}_2, \bar{v}_3, \cdots, \bar{v}_k, \cdots]\otimes Z_p,$$

where $Z_p=Z/pZ$, which contains $Z_p[u_1, \bar{v}_2, \bar{v}_3, \cdots, \bar{v}_k, \cdots]$. Thus we obtain the proof of Theorem 3 since $\bar{v}_1\equiv u_1 \bmod p$.

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